

3.7

Related Rates

In this section we look at problems that ask for the rate at which some variable changes. In each case the rate is a derivative that has to be computed from the rate at which some other variable (or perhaps several variables) is known to change. To find it, we write an equation that relates the variables involved and differentiate it to get an equation that relates the rate we seek to the rates we know. The problem of finding a rate you cannot measure easily from some other rates that you can is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3} \pi r^3.$$

Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to measure directly the rate of increase of the volume than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt .

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

EXAMPLE 1 Pumping Out a Tank

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Solution We draw a picture of a partially filled vertical cylindrical tank, calling its radius r and the height of the fluid h (Figure 3.42). Call the volume of the fluid V .

As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. We are told that

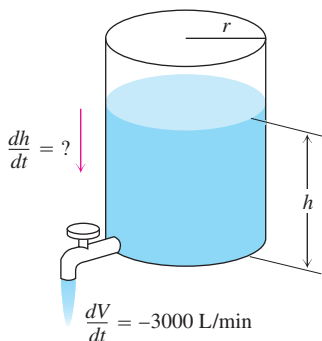


FIGURE 3.42 The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

$$\frac{dV}{dt} = -3000.$$

We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.

We are asked to find

$$\frac{dh}{dt}.$$

How fast will the fluid level drop?

To find dh/dt , we first write an equation that relates h to V . The equation depends on the units chosen for V , r , and h . With V in liters and r and h in meters, the appropriate equation for the cylinder's volume is

$$V = 1000 \pi r^2 h$$

because a cubic meter contains 1000 L.

Since V and h are differentiable functions of t , we can differentiate both sides of the equation $V = 1000\pi r^2 h$ with respect to t to get an equation that relates dh/dt to dV/dt :

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt} \quad r \text{ is a constant.}$$

We substitute the known value $dV/dt = -3000$ and solve for dh/dt :

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}.$$

The fluid level will drop at the rate of $3/(\pi r^2)$ m/min.

The equation $dh/dt = -3/\pi r^2$ shows how the rate at which the fluid level drops depends on the tank's radius. If r is small, dh/dt will be large; if r is large, dh/dt will be small.

$$\text{If } r = 1 \text{ m: } \frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95 \text{ m/min} = -95 \text{ cm/min.}$$

$$\text{If } r = 10 \text{ m: } \frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 \text{ m/min} = -0.95 \text{ cm/min.} \quad \blacksquare$$

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants.* Use t for time. Assume that all variables are differentiable functions of t .
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t .* Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

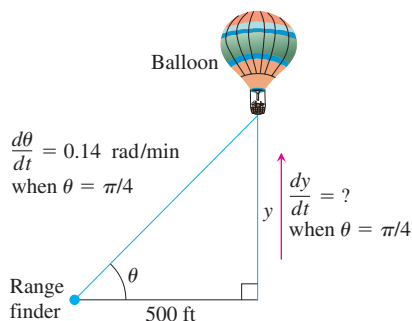


FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 2 A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.43). The variables in the picture are
 θ = the angle in radians the range finder makes with the ground.
 y = the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t .

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \quad \text{when} \quad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$.

4. Write an equation that relates the variables y and θ .

$$\frac{y}{500} = \tan \theta \quad \text{or} \quad y = 500 \tan \theta$$

5. Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500 (\sec^2 \theta) \frac{d\theta}{dt}$$

6. Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt .

$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = 140 \quad \sec \frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■

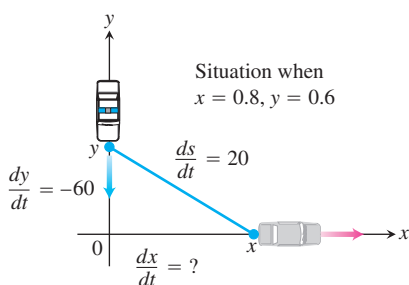


FIGURE 3.44 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x -axis as the eastbound highway and the positive y -axis as the southbound highway (Figure 3.44). We let t represent time and set

$$\begin{aligned} x &= \text{position of car at time } t \\ y &= \text{position of cruiser at time } t \\ s &= \text{distance between car and cruiser at time } t. \end{aligned}$$

We assume that x , y , and s are differentiable functions of t .

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right). \end{aligned}$$

Finally, use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt .

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}} \left(0.8 \frac{dx}{dt} + (0.6)(-60) \right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ■

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 3.45 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of water in tank at time t .

We assume that V , x , and y are differentiable functions of t . The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3} \pi x^2 y.$$

This equation involves x as well as V and y . Because no information is given about x and dx/dt at the time in question, we need to eliminate x . The similar triangles in Figure 3.45 give us a way to express x in terms of y :

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3} \pi \left(\frac{y}{2} \right)^2 y = \frac{\pi}{12} y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt .

$$9 = \frac{\pi}{4} (6)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

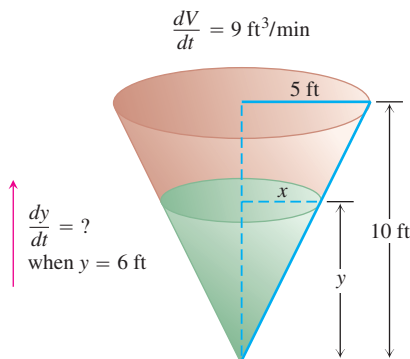


FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 4).