الف) انتگرال های زیر را با روش تغییر متغیر حل کنید.

1) $\int \frac{x+1}{2\sqrt{x+1}}dx =$

2) $\int x^2\sqrt{x+1}dx =$

3) $\int x^3\sqrt{x^2-1}dx =$

4) $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx =$

5) $\int \frac{x+1}{(x^2+2x+2)^3}dx =$

6) $\int \frac{x^2+1}{\sqrt[3]{x^3+3x+1}}dx =$

7) $\int x^2\sqrt[3]{1-x}\,dx =$

8) $\int x\sqrt[5]{5-x^2}\,dx =$

9) $\int sin^2x\,.\,cos^3x\,dx =$

10) $\int \frac{sinx}{cos^2x+2cosx+1}dx =$

11) $\int \frac{cos^3 3x}{\sqrt[3]{sin3x}}dx =$

12) $\int \sqrt{1+sin^2(x-1)}\,.sin(x-1).cos(x-1)dx =$

13) $\int \frac{1+sin3x}{cos^2 3x}dx =$

14) $\int \frac{(sinx+cosx)}{(sinx-cosx)^{\frac{1}{3}}}dx =$

15) $\int \frac{sin^3x}{\sqrt[3]{cos^5x}}dx =$

16) $\int sin2x\sqrt{1+sin^2x}\,dx =$

17) $\int \frac{(2x+5)^7}{(3x-1)^9}dx =$

18) $\int \sqrt{(x^2-x^{-2})^2+4}\;dx =$

19) $\int \frac{2e^x}{1+e^x}dx =$

20) $\int (x+1)\sqrt[3]{x-3}\,dx =$

21) $\int x\,(x-2)^{53}\,dx =$

22) $\int \frac{cos\sqrt{x}}{\sqrt{x}}dx =$

23) $\int \frac{Lnx}{x}dx =$

24) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx =$

25) $\int \sqrt{a^2-x^2}\,dx =?\quad , \quad 0\le x\le \frac{\pi}{2}\quad ,(Hint: Let\ x=a\ sin\theta)$

26) $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}dx =$

27) $\int \sqrt{\tan x}\;sec^2x\,dx =$

28) $\int cos^5x\,dx =$

29) $\int \frac{cos\sqrt{x}}{\sqrt{x}\;sin^2\sqrt{x}}dx$

30) $\int (\frac{1}{x^2})\ sin^5(\frac{1}{x})\ \cos(\frac{1}{x})dx =$

31) $\int \frac{x\ cos\sqrt{3x^2-6}}{\sqrt{3x^2-6}}dx$

32) $\int \sec x\ dx =$

33) $\int\ e^{\frac{1}{x}}\ (\frac{1}{x^2})dx =$

34) $\int \frac{1}{x\ Lnx}dx =$

35) $\int \frac{\sin(Lnx)}{x}dx =$

36) $\int \frac{e^{2\sqrt{x+3}}}{\sqrt{x+3}}dx =$

37) $\int \frac{dx}{\sqrt{1-x^2}\ (sin^{-1}x)^3} =$

38) $\int \frac{\cos(Lnx^3)dx}{x} =$

39) $\int \frac{3^{Lnx}}{x}dx =$

40) $\int \frac{e^{2x}}{\sqrt{e^x+1}}dx =$

41) $\int \frac{2^x}{1+4^{2x}}dx =$

42) $\int \frac{sec^2x}{9+tan^2x}dx =$

43) $\int \frac{1+Lnx}{5+xLnx}dx =$

44) $\int_{-\infty}^{0} \frac{1}{1+e^{-x}} dx =$

45) $\int_{1}^{2} (2+e^{3x})^2 \, dx =$

46) $\int_{1}^{\sqrt{e}} \frac{1}{x\sqrt{1-(Lnx)^2}} dx =$

47) $\int_{e^2}^{e^4} \frac{1}{xLnx\,(Ln(Lnx))} dx =$

48) $\int \frac{4(Lnx)^3+3}{x\,((Lnx)^4+3Lnx)} dx =$

ب) انتگرال های زیر را با روش جزء به جزء حل کنید.

49) $\int x^2 e^x \, dx =$

50) $\int (3x+2)^2 e^{2x-5} \, dx =$

51) $\int xsinx \, dx =$

52) $\int e^x \, sinx \, dx =$

53) $\int e^x \, cosx \, dx =$

54) $\int e^{ax} \, sinbx \, dx =$

55) $\int e^{ax} \, cosbx \, dx =$

56) $\int Ln\, x \, dx =$

57) $\int x^2 Ln\, x \, dx =$

58) $\int \sin\,(Lnx) \, dx =$

59) $\int x\, 2^x\, dx =$

60) $\int tan^{-1}x\ dx =$

61) $\int sec^3 x\, dx =$

ج) انتگرال های زیر را با تجزیه به کسرهای ساده حل کنید.

62) $\int \frac{3x^2-5x+1}{(x-2)(x+3)(x^2-1)}dx =$

63) $\int \frac{x^2+x+3}{(x+2)(x^2+1)}dx =$

حل این مسایل در زیر آمده است و در آینده ویرایش خواهد شد.

1) $\int \frac{x+1}{2\sqrt{x+1}}\ dx = \frac{1}{2}\int \sqrt{x+1}\, dx = \frac{1}{2}\int \sqrt{u}\, du$

$$= \frac{1}{2}\times\ \frac{2}{3}\ u^{\frac{3}{2}} + c = \frac{1}{3}\ (x+1)^{\frac{3}{2}} + c$$

توضیح: مي دانيم $(x+1)=(\sqrt{x+1})(\sqrt{x+1})$ فرض مي كنيم: $x+1=u \Rightarrow dx=du$

2) $\int x^2\ \sqrt{1+x}\, dx = \int (u-1)^2\ \sqrt{u}\, du$

$$= \int (u^2-2u+1)u^{\frac{1}{2}}\, du$$

$$= \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\, du$$

$$= \frac{2}{7}u^{\frac{7}{2}} + (-2)\left(\frac{2}{5}\right)u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + c$$

3) $\int x^3\sqrt{x^2-1}\,dx = \int x^2\sqrt{x^2-1}(x\,dx) = \int (u+1)\sqrt{u}(\frac{1}{2}du)$

$= \frac{1}{2}\int (u^{\frac{3}{2}}+u^{\frac{1}{2}})du = (\frac{1}{2})(\frac{2}{5})u^{\frac{5}{2}} + (\frac{1}{2})(\frac{2}{3})u^{\frac{3}{2}} + c$

$= \frac{1}{5}(x^2-1)^{\frac{5}{2}} + \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c$

توضیح: $x^2-1=u \Rightarrow 2xdx = du$

4) $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx = \int u^2(2du) = 2\frac{u^3}{3} + c = \frac{2}{3}\left(\sqrt{x}-1\right)^3 + c$

توضیح: $\sqrt{x}-1=u \Rightarrow \frac{dx}{2\sqrt{x}} = du$

5) $\int \frac{(x+1)dx}{(x^2+2x+2)^3} = \int \frac{\frac{1}{2}du}{u^3} = \left(\frac{1}{2}\right)\frac{u^{-2}}{-2} + c = \frac{-1}{4}(x^2+2x+2)^{-2} + c$

توضیح: $x^2+2x+2=u \Rightarrow 2(x+1)dx = du$

6) $\int \frac{x^2+1}{\sqrt[3]{x^3+3x+1}}\,dx = \int \frac{\frac{1}{3}du}{\sqrt[3]{u}} = \frac{1}{3}\int u^{\frac{-1}{3}}du = \left(\frac{1}{3}\right)\left(\frac{3}{2}\right)u^{\frac{2}{3}} + c$

$= \frac{1}{2}(x^3+3x+1)^{\frac{2}{3}} + c$

توضیح: $x^3+3x+1=u \Rightarrow (3x^2+3)dx = du$

7) $\int x^2\sqrt[3]{1-x}dx = \int (1-u)^2\sqrt[3]{u}(-du) = -\int\left(u^{\frac{1}{3}} - 2u^{\frac{4}{3}} + u^{\frac{7}{3}}\right)du$

$= \frac{-3}{4}u^{\frac{4}{3}} + (2)u^{\frac{7}{3}}\left(\frac{3}{7}\right) - \frac{3}{10}u^{\frac{10}{3}} + c$

$= \frac{-3}{4}(1-x)^{\frac{4}{3}} + \frac{6}{7}(1-x)^{\frac{7}{3}} - \frac{3}{10}(1-x)^{\frac{10}{3}} + c$

توضیح: $1-x=u \Rightarrow du=-dx \quad , \quad x = 1-u$

8) $\int x\sqrt[5]{5-x^2}dx = \int \frac{-1}{2}u^{\frac{1}{5}}du = \frac{-1}{2}\left(\frac{5}{6}\right)u^{\frac{6}{5}} + c$

$$= \frac{-5}{12}(5 - x^2)^{\frac{6}{5}} + c$$

توضیح: $1 - x^2 = u \Rightarrow -2xdx = du$

9) $\int \sin^2 x . \cos^3 x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$$= \int z^2 (1 - z^2) dz = \frac{1}{3} z^3 - \frac{1}{5} z^5 + c$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$$

10) $\int \frac{\sin x \, dx}{\cos^2 x + 2\cos x + 1} = \int \frac{\sin x dx}{(\cos x + 1)^2} = \int \frac{-du}{u^2} = \frac{1}{u} + c = \frac{1}{\cos x + 1} + c$

توضیح: $\cos x + 1 = u \Rightarrow -\sin x dx = du$

11) $\int \frac{\cos^3 3x}{\sqrt[3]{\sin 3x}} dx = \int \frac{(1 - \sin^2 3x) \cos 3x \, dx}{\sqrt[3]{\sin 3x}} = \int \frac{\frac{1}{3}(1 - u^2) du}{\sqrt[3]{u}}$

$$= \frac{1}{3} \int \left(u^{\frac{-1}{3}} - u^{\frac{5}{3}} \right) du = \left(\frac{1}{3}\right) \left(\frac{3}{2}\right) u^{\frac{2}{3}} - \left(\frac{1}{3}\right) \left(\frac{3}{8}\right) u^{\frac{8}{3}} + c$$

$$= \frac{1}{2} (\sin 3x)^{\frac{2}{3}} - \frac{1}{8} (\sin 3x)^{\frac{8}{3}} + c$$

توضیح: $\sin 3x = u \Rightarrow 3 \cos 3x dx = du$

12) $\int \sqrt{1 + \sin^2(x-1)} . \sin(x-1) . \cos(x-1) \, dx = \int \sqrt{u} \left(\frac{1}{2}\right) du$

$$= \left(\frac{1}{2}\right) \left(\frac{2}{3}\right) u^{\frac{3}{2}} + c = \frac{1}{3} (1 + \sin^2(x-1))^{\frac{3}{2}} + c$$

توضیح: $1 + \sin^2(x-1) = u \Rightarrow du = 2\sin(x-1) . \cos(x-1) dx$

13) $\int \frac{1 + \sin 3x}{\cos^2 3x} dx = \int \frac{dx}{\cos^2 3x} + \int \frac{\sin 3x dx}{\cos^2 3x}$

$$= \frac{1}{3} \tan 3x + \frac{-1}{3} \int \frac{du}{u^2}$$

$$= \frac{1}{3} \tan 3x + \frac{1}{3u} + c$$

$$= \frac{1}{3}\tan 3x + \frac{1}{3\cos 3x} + c$$

تذکر: $(\tan 3x)' = 3(1+\tan^2 3x) = \frac{3}{\cos^2 3x}$ و $\cos 3x = u \Rightarrow -3\sin 3x dx = du$

14) $\int \frac{(\sin x + \cos x)dx}{(\sin x - \cos x)^{\frac{1}{3}}} = \int \frac{du}{u^{\frac{1}{3}}} = \frac{3}{2}u^{\frac{2}{3}} + c = \frac{3}{2}(\sin x - \cos x)^{\frac{2}{3}} + c$

15) $\int \frac{\sin^3 x dx}{\sqrt[3]{\cos^5 x}} = \int \frac{(1-\cos^2 x)\sin x dx}{(\cos x)^{\frac{5}{3}}} = \int -(1-u^2)u^{\frac{-5}{3}} du$

$$= \int \left(-u^{\frac{-5}{3}} + u^{\frac{1}{3}}\right) du = \frac{3}{2}u^{\frac{-2}{3}} + \frac{3}{4}u^{\frac{4}{3}} + c$$

$$= \frac{3}{2\sqrt[3]{\cos^2 x}} + \frac{3}{4}(\cos x)^{\frac{4}{3}} + c$$

توضیح: $\cos x = u \Rightarrow -\sin x\, dx = du$

16) $\int \sin 2x \sqrt{1+\sin^2 x}\, dx = \int \sqrt{u} du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+\sin^2 x)^{\frac{3}{2}} + c$

توضیح: $1 + \sin^2 x = u \Rightarrow du = 2\sin x \cos x\, dx \Rightarrow du = \sin 2x\, dx$

17) $\int \frac{(2x+5)^7}{(3x-1)^9}\, dx = \int (\frac{2x+5}{3x-1})^7 \frac{1}{(3x-1)^2}\, dx$

$$= \int u^7 \left(\frac{-1}{17}\right) du = \left(\frac{-1}{17}\right)\frac{u^8}{8} + c$$

$$= \frac{-1}{136}(\frac{2x+5}{3x-1})^8 + c$$

توضیح: $\frac{2x+5}{3x-1} = u \Rightarrow \frac{-17}{(3x-1)^2}\, dx = du$

18) $\int \sqrt{(x^2 - x^{-2})^2 + 4}\, dx = \int \sqrt{x^4 + 2 + x^{-4}}\, dx$

$$= \int \sqrt{(x^2 + x^{-2})^2}\, dx$$

$$= \int (x^2 + x^{-2})dx = \frac{1}{3}x^3 - \frac{1}{x} + c$$ ، با فرض $x > 0$

19) $\int \frac{2e^x}{1+e^x}\,dx = 2\int \frac{du}{u} = 2lnu + c = 2\ln(1+e^x) + c$

20) $\int (x+1)\sqrt[3]{x-3}dx = \int (u+4)\sqrt[3]{u}du = \int \left(u^{\frac{4}{3}} + 4u^{\frac{1}{3}}\right) du = \frac{3}{7}u^{\frac{7}{3}} + (4)(\frac{3}{4})u^{\frac{4}{3}} + c = \frac{3}{7}(x-3)^{\frac{7}{3}} + 3(x-3)^{\frac{4}{3}} + c$

$x - 3 = u \Rightarrow dx = du, x + 1 = u + 4$

21) $\int x(x-2)^{53}dx = \int u^{53}(u+2)du = \int (u^{54} + 2u^{53})du = \frac{1}{55}u^{55} + \frac{2}{54}u^{54} + c = \frac{1}{55}(x-2)^{55} + \frac{1}{27}(x-2)^{54} + c$

$x - 2 = u \Rightarrow x = (u+2)$

22) $\frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cos u\,du = 2\sin u + c = 2\sin\sqrt{x} + c$

$\sqrt{x} = u \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

23) $\int \frac{lnx}{x}dx = \int udu = \frac{u^2}{2} + c = \frac{1}{2}(lnx)^2 + c$

24) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{du}{u} = lnu + c = \ln(e^x + e^{-x}) + c$

$e^x + e^{-x} = u \Rightarrow du = (e^x - e^{-x})dx$

25) $\int \sqrt{a^2 - x^2}\,dx = \int \sqrt{a^2\cos^2\theta}\,a\cos\theta\,d\theta = \int a^2\cos^2\theta\,d\theta = a^2\int \frac{1-\cos 2\theta}{2}\,d\theta = (\theta - \frac{1}{2}\sin 2\theta) + c = a^2(\sin^{-1}\frac{x}{a} - \frac{1}{2}\sin 2\theta) + c$

بافرض $0 \le x \le \frac{\pi}{2} \Rightarrow x = a\sin\theta$

26) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2} = 2\int \frac{du}{u^2} = \frac{-2}{u} + c = \frac{-2}{1+\sqrt{x}} + c$

$1 + \sqrt{x} = u \Rightarrow du = \frac{1}{2\sqrt{x}}dx$

27)$\int \sqrt{\tan x}\,.sec^2xdx = \int \sqrt{u}.du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(\tan x)^{\frac{3}{2}} + c$

28)$\int cos^5xdx = \int cos^4x.\cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx =$

$$\int (1-u^2)^2du = \int (1-2u^2+u^4)du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c =$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

29)$\int \frac{\cos\sqrt{x}}{\sqrt{x}\sin^2\sqrt{x}}\,dx = 2\int \frac{du}{u^2} = \frac{-2}{u} + c = \frac{-2}{\sin\sqrt{x}} + c$

$$\sin\sqrt{x} = u \Rightarrow du = \frac{\cos\sqrt{x}dx}{2\sqrt{x}} = du$$

30)$\int (\frac{1}{x^2})\sin^5\frac{1}{x}\cos\frac{1}{x}\,dx = -\int u^5du = -\frac{u^6}{6} + c = \frac{-1}{6}\sin^6\frac{1}{x} + c$

$$\sin\frac{1}{x} = u \Rightarrow du = \frac{-1}{x^2}\cos\frac{1}{x}\,dx$$

31)$\frac{x\cos\sqrt{3x^2-6}}{\sqrt{3x^2-6}}\,dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + c = \frac{1}{3}\sin\sqrt{3x^2-6} + c$

$$\sqrt{3x^2-6} = u \Rightarrow du = \frac{6xdx}{2\sqrt{3x^2-6}}$$

32)$\int secxdx = \int \frac{secx(secx+\tan x)}{(secx+\tan x)}\,dx = \int \frac{du}{u} = lnu + c$

$$= \ln\,|secx + \tan x| + c$$

راهنمایی:در $(secx + \tan x)$ضرب و تقسیم کنید

زیرا $(secx + \tan x) = u \Rightarrow du = (\tan x.secx + sec^2x)dx$

33)$\int e^{\frac{1}{x}}.\frac{1}{x^2}\,dx = \int e^u(-du) = -e^u + c = -e^{\frac{1}{x}} + c$

34)$\int \frac{dx}{xlnx} = \int \frac{du}{u} = lnu + c = \ln(lnx) + c$

35) $\int \frac{\sin(lnx)}{x}\,dx = \int \sin u\,du = -\cos u + c = -\cos(lnx) + c$

36) $\int \frac{e^{2\sqrt{x+3}}}{\sqrt{x+3}}\,dx = \int e^{u}du = e^{u} + c = e^{2\sqrt{x+3}} + c$

$2\sqrt{x+3} = u \Rightarrow 2\frac{dx}{2\sqrt{x+3}} = du$

ادامه مسائل انتگرال + حل

1) $\int e^x (f(x) + f'(x))\, dx = f(x).e^x + c$

2) $\int \frac{x\, e^x}{(1+x)^2} dx = \frac{e^x}{x+1} + c$

3) $\int Ln(a^2 + x^2)\, dx = x\, Ln(a^2 + x^2) - \int \frac{x}{a^2 + x^2} dx$

$$\Rightarrow x Ln(a^2 + x^2) - \frac{1}{2} Ln(a^2 + x^2) + c$$

4) $\int x^3 e^{-x^2}\, dx = -\frac{1}{2}\int u\, e^u\, du = -\frac{1}{2}(-x^2 - 1) e^{-x^2} + c$

5) $\int x^5 e^x\, dx = (x^5 - 5x^4 + 2\circ x^3 - 6\circ x^2 + 12\circ x - 12\circ)\, e^x$

6) $\mathrm{I} = \int \sin(Lnx)\, dx$

$u = \sin(Lnx) \quad , dv = dx \Rightarrow du = \frac{1}{x}\cos(Lnx) \quad , v = x$

$\mathrm{I} = x\sin(Lnx) - \int \cos(Lnx)\, dx$

$u = \cos(Lnx) \quad , dv = dx \Rightarrow du = -\frac{1}{x}\sin(Lnx) \,, v = x$

$\mathrm{I} = x\sin(Lnx) - x\cos(Lnx) - \mathrm{I}$

$\mathrm{I} = \frac{x}{2}(\sin(Lnx) - \cos(Lnx)) + c$

7) $\mathrm{I} = \int x\, tg^{-1}x\, dx$

$u = tg^{-1}x, dv = x\, dx \Rightarrow dv = \frac{1}{1+x^2} dx \quad , \quad v = \frac{x^2}{2}$

$\mathrm{I} = \frac{x^2}{2} tg^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1} dx = \frac{x^2}{2} tg^{-1}x - \frac{1}{2}x + \frac{1}{2} tg^{-1}x + c$

8) $\int \sin^{-1}\sqrt{x}\ dx \qquad t^2 = x \Rightarrow 2tdt = dx$

$$I = 2\int t\sin^{-1} t dt = t^2 \sin^{-1} t - \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \int \frac{1-t^2+1}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \sin^{-1} t + \int \sqrt{1-t^2}\ dt$$

$$\int \sqrt{1-t^2}\ dt = \int \cos^2\theta\, d\theta = \frac{1}{2}t + \frac{1}{4}\cos 2\theta$$

$$\Rightarrow I = x\sin^{-1}\sqrt{x} + \sin^{-1}\sqrt{x} + \frac{1}{2}\cos^{-1}\sqrt{x} + \frac{1}{4}\cos 2(\cos^{-1}\sqrt{x})$$

9) $I = \int Ln(x+\sqrt{1+x^2})\,dx$

$$u = Ln(x+\sqrt{1+x^2}), dv = dx \Rightarrow du = \frac{1}{\sqrt{1+x^2}}, v = x$$

$$I = x\ Ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} = x\ Ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + c$$

10) $I = \int x\ Ln(\frac{1+x}{1-x})\,dx$

$$u = Ln(\frac{1+x}{1-x}), dv = dx \Rightarrow du = \frac{2}{1-x^2}dx \qquad , v = x$$

$$I = x\ Ln(\frac{1+x}{1-x}) - \int \frac{2x}{1-x^2}dx = x\ Ln(\frac{1+x}{1-x}) + Ln(1-x^2) + c$$

11) $I = \int e^{\sqrt{x}}\ dx$

$$u^2 = x \Rightarrow 2udu = dx$$

$$I = 2\int u\,e^u\ du = 2(u-1)e^u + c = 2(\sqrt{x}-1)e^{\sqrt{x}} + c$$

12)

$$\mathrm{I} = \int x^2 \ Ln(\frac{1+x}{1-x})\,dx$$

$$u = Ln(\frac{1-x}{1+x}) \quad , \qquad dv = x^2\,dx \qquad \Rightarrow du = -\frac{2}{1-x^2}dx \qquad , v = \frac{x^3}{3}dx$$

$$\mathrm{I} = \frac{x^3}{3}Ln(\frac{1-x}{1+x}) + \frac{1}{3}\int x^3 1 - x^2\,dx$$

$$= \frac{x^3}{3}Lx(\frac{1-x}{1+x}) - \frac{1}{3}\int (x + \frac{x}{x^2-1})\,dx$$

$$= \frac{x^3}{3}Lx(\frac{1-x}{1+x}) - \frac{1}{3}x^2 - \frac{1}{6}Ln(x^2-1) + c$$

13)

$$\mathrm{I} = \int \frac{tg^{-1} e^x}{e^x}\,dx$$

$$u = tg^{-1} e^x \qquad , dv = \frac{dx}{e^x} \Rightarrow du = \frac{e^x}{1+e^{2x}}dx\,, v = -\frac{1}{e^x}$$

$$\mathrm{I} = -\frac{tg^{-1} e^x}{e^x} + \int \frac{dx}{1+e^{2x}} = \frac{tg^{-1} e^x}{e^x} + \int \frac{du}{u(1+u^2)}$$

$$= -\frac{tg^{-1} e^x}{e^x} + Ln\ e^x + \frac{1}{2}\ Ln(1+e^{2x}) - 2tg^{-1} e^x + c$$

14)

$$\mathrm{I} = \int (\sin^{-1} x)^2\,dx$$

$$u = (\sin^{-1} x)^2 \quad , dv = dx \quad \Rightarrow du = \frac{2}{\sqrt{1-x^2}}\sin^{-1} x\ dx\ , v = x$$

$$\mathrm{I} = x(\sin^{-1} x)^2 - \int \frac{2}{\sqrt{1-x^2}}\sin^{-1} x\ dx$$

$$= x(\sin^{-1} x)^2 + 2(\sqrt{1-x^2})\sin^{-1} x - 2x + c$$

15)

$$\mathrm{I} = \int \frac{Ln(1+x)}{2(1+x)}\,dx = \frac{1}{2}\int u\ du$$

$$= \frac{1}{4}(Ln(1+x))^2 + c$$

16) $I=\int (\frac{Lnx}{x})^2\,dx$

$u=(\frac{Lnx}{x})^2 \quad , \qquad dv=dx \qquad \Rightarrow du=2(\frac{1-Lnx}{x^2})(\frac{Lnx}{x})\,dx\,,\, v=x$

$I=x(\frac{Lnx}{x})^2-2\int \frac{Lnx-(Lnx)^2}{x^2}\,dx$

$=x(\frac{Lnx}{x})^2-2\int \frac{Lnx}{x^2}dx+2\int (\frac{Lnx}{x})^2\,dx$

$-I=x(\frac{Lnx}{x})^2+2\int \frac{1-Lnx-1}{x^2}\,dx$

$I=-x(\frac{Lnx}{x})^2-2(\int (\frac{Lnx}{x})'\,dx-\int \frac{1}{x^2}dx)+c$

$=-x(\frac{Lnx}{x})^2-2(\frac{Lnx}{x})+\frac{1}{x}+c$

17) $I=\int_0^{\frac{\pi^2}{2}} \cos\sqrt{2x}\,dx$

$u=\sqrt{2x} \qquad \Rightarrow \qquad 2udu=2dx$

$I=\int_0^{\pi} u\,\cos\,u\,\,du=u\,\sin\,u+\cos u+c$

18) $I=\int_1^4 \sec^{-1}\sqrt{x}\,\,dx \quad , \qquad u=\sqrt{x}$

$2u\,du=dx$

$I=2\int_1^2 u\,\sec^{-1}u\,\,du=2\int_1^2 u\,\cos^{-1}(\frac{1}{u})\,du$

$t=\cos^{-1}(\frac{1}{u})\Rightarrow dt=-\frac{1}{u^2}.\frac{-1}{\sqrt{1-\frac{1}{u^2}}}\,du=\frac{1}{u\sqrt{u^2-1}}du$

$I=2(\frac{u^2}{2}\cos^{-1}(\frac{1}{u}))-\int \frac{u}{\sqrt{u^2-1}}\,du$

$=u^2\cos^{-1}(\frac{1}{u})-\sqrt{u^2-1}\,\Big|_1^2=4\times\frac{\pi}{3}-\sqrt{3}$

19) $$I=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x.\cot x.\csc x\,dx=-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x(\csc x)'\,dx$$

$$u=x \quad , \quad dv=(\csc x)'dx \Rightarrow du=dx\ ,\ v=\csc x$$

$$I=x\csc x\Bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\csc x\,dx$$

$$\int\csc x\,dx=\int\frac{dx}{\sin x}=\int\frac{\sin x}{1-\cos^2 x}dx=\frac{1}{2}Ln\left|\frac{1-\cos x}{1+\cos x}\right|$$

$$I=(\frac{3\pi}{4}\times\sqrt{2}-\frac{\pi}{4}\times\sqrt{2})-\frac{1}{2}(Ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|-Ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|)$$

20) $$I=\int_{0}^{\frac{\pi^2}{4}}\sin\sqrt{x}\,dx \qquad u^2=x\Rightarrow 2u\,du=dx$$

$$I=2\int_{0}^{\frac{\pi}{2}}u\sin u\,du=2(-u\cos u+\sin u)\Bigg|_{0}^{\frac{\pi}{2}}$$

21) $$\int\frac{xdx}{\sqrt{1+x^2\sqrt{(1+x^2)^3}}}=\int\frac{xdx}{\sqrt{1+x^2(1+x^2)\sqrt{1+x^2}}}$$

$$u^2=1+x^2\Rightarrow udu=xdx$$

$$\int\frac{udu}{\sqrt{1+(u^2-1)u^3}}=\int\frac{udu}{\sqrt{1-u^3+u^5}}$$

22) $$\int\frac{\sqrt{4-x^2}}{x^4}dx;u=2\sin x\Rightarrow du=2\cos xdx$$

$$\frac{1}{4}\int\frac{\cos^2 x}{\sin^4 x}dx=\frac{1}{4}\int\cot^2 x\cdot\csc^2 xdx=-\frac{1}{12}\cot^3 x+c$$

23) $$\int\frac{dx}{\cos^2 x\ .\ \sin^2 x}=4\int\frac{dx}{(2\sin x\cos x)^2}=4\int\frac{dx}{\sin^2 2x}=-2\cot 2x+c$$

24) $$\int \tan^4 x dx = \int \tan^2 x(\sec^2 x - 1)dx$$
$$= \int \tan^2 x \sec^2 x dx - \int (\tan^2 x + 1)dx + \int dx$$
$$= \frac{\tan^3 x}{3} - \tan x + x + c$$

25) $$\int \frac{\cos^3 x}{\sin^9 x} dx$$
$$= \int \frac{\cos x(1-\sin^2 x)}{\sin^9 x} dx = \int \frac{\cos x}{\sin^9 x} dx - \int \frac{\cos x}{\sin^7 x} dx$$
$$= -\frac{1}{8\sin^8 x} + \frac{1}{6\sin^6 x} + c$$

26) $$\int \sec^4 x \cdot \cot^6 x dx = \int \frac{\sec^4 x}{\tan^6 x} dx = \int \frac{\sec^2 x(1+\tan^2 x)}{\tan^6 x} dx$$
$$= \int \frac{\sec^2 x}{\tan^6 x} dx + \int \frac{\sec^2 x}{\tan^4 x} dx = -\frac{1}{5\tan^5 x} - \frac{1}{3\tan^3 x} + c$$

27) $$\int \frac{\sin x\, dx}{\cos^2 x + 2\cos + 1} = \int \frac{\sin x\, dx}{(\cos x + 1)^2}$$
$$(u = \cos x + 1 \quad \Rightarrow \quad -du = \sin x\, dx)$$
$$= -\int \frac{du}{u^2} = \frac{1}{\cos x + 1} + c$$

28) $$\int \frac{(\sin x + \cos x)}{\sqrt[3]{(\sin x - \cos x)}} \qquad u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x)_{dx}$$

$$= \int \frac{dy}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}} du = \frac{2}{3} u^{\frac{2}{3}} + c = \frac{2}{3}(\sin x - \cos x)^{\frac{2}{3}} + c$$

29) $In=\int \tan^{n-2}x\,.\,\tan^2 x\,dx=\int \tan^n x\left(\sec^2 x-1\right)dx=\tan x$

$$=\int \tan^{n-1}x\,.\,\tan^2 x\,dx-\int \tan^{n-2}x\left(\sec^{n-2}x-1\right)dx$$

$$=\frac{\tan^{n-1}}{n-1}-I_{n-2}6+c$$

$$\Rightarrow\qquad In+I_{n-2}=\frac{\tan^{n-1}}{n-1}+c$$

$$I_2=\int \tan^3 x\,dx=\int\left(\sec^2 x-1\right)dx=\tan x-x$$

$$I_4=\frac{\tan^3 x}{3}-\tan x+x+c$$

$$I_6=\frac{\tan^5 x}{5}-\frac{\tan^3 x}{3}+\tan x-x+c$$

30) $\int \sin^4 x\cos^4 x\,dx=\frac{1}{16}\int(2\sin x\cos x)^4\,dx$

$$=\frac{1}{16}\int \sin^4 2x\;dx=\frac{1}{32}\int(1-\cos 4x)^2$$

$$=\frac{1}{32}\int\left(1-2\cos 4x+\cos^2 4x\right)dx$$

$$=\frac{1}{32}\left(x-\frac{1}{2}\sin 4x+\frac{1}{2}x+\frac{1}{16}\sin 8x\right)+c$$

31) فرض کنید f بر $[-a\ ,\ a]$ پیوسته و فرد باشدو ثابت کنید: $\int_{-a}^{a} f(x)\,dx=0$

$$\int_{-a}^{a} f(x)\,dx=\int_{-a}^{0} f(x)\,dx+\int_{0}^{a} f(x)\,dx$$

$$=\int_{a}^{0} f(-x)\,(-dx)+\int_{0}^{a} f(x)\,dx$$

$$=\int_{0}^{a} f(-x)\,dx+\int_{0}^{a} f(x)\,dx$$

$$=-\int_{0}^{a} f(x)\,dx+\int_{0}^{a} f(x)\,dx=0$$

32) اگر f زوج باشد, ثابت کنید که:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

33) $I = \int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$

اگر قرا دهیم $u = \frac{\pi}{2} - x$ پس $du = -dx$ و داریم

$$I = \int_{\frac{\pi}{2}}^{0} \frac{\sqrt{\sin}}{\sqrt{\sin u} + \sqrt{\cos u}}(-du) = \int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\sin u}}{\sqrt{\cos u} + \sqrt{\sin u}}\,du$$

$$I + I = 2I = \int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\cos u}}{\sqrt{\cos u} + \sqrt{\sin u}}\,du + \int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\cos u}}{\sqrt{\cos u} + \sqrt{\sin u}}\,du$$

$$= \int_{0}^{\frac{\pi}{2}} du = \frac{\pi}{2} \qquad \Rightarrow \qquad I = \frac{\pi}{2}$$

34) $I = \int_{0}^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x}\,dx$

حل) اگر قرار دهیم $u = \frac{\pi}{2} - x$ پس $du = -dx$ و داریم:

$$I = -\int_{\frac{\pi}{2}}^{0} \frac{\cos^m u}{\cos^m u + \sin^m u}\,dx = \int_{0}^{\frac{\pi}{2}} \frac{\cos^m u}{\cos^m u + \sin^m u}\,du$$

$$I + I = 2I = \int_{0}^{\frac{\pi}{2}} \frac{\sin^m u}{\sin^m u + \cos^m u}\,du + \int_{0}^{\frac{\pi}{2}} \frac{\cos^m u}{\cos^m u + \sin^m u}\,du = \int_{0}^{\frac{\pi}{2}} du = \frac{\pi}{2}$$

$$\Rightarrow \qquad I = \frac{\pi}{4}$$

35) $\int_{-2}^{2} \ln(x+\sqrt{1+x^2})dx = \circ$

$$\int_{-2}^{2} \ln(x+\sqrt{1+x^2})dx = \circ$$

چون تابع زیر انتگرال فرد است:

$$f(-x)\ln(-x+\sqrt{x^2+1}) = \ln\frac{1}{x+\sqrt{x^2+1}} = -\ln(x=\sqrt{x^2+1}) = -f(x)$$

36) $I = \int_{e^2}^{e^4} \frac{dx}{x\ln x(\ln(\ln x))}$

$$u = \ln(\ln x) \;\Rightarrow\; du = \frac{dx}{x\ln x}$$

$$\Rightarrow\; I = \int_{\ln 2}^{\ln 4} \frac{du}{u} = \ln u \Big|_{\ln 2}^{\ln 4} = \ln(\ln(\ln 4) - \ln(\ln 2)$$

37) $\int \frac{1+\ln x}{5+x\ln x}dx$ $\qquad u = 5 + x\ln x \Rightarrow du = (1+\ln x)dx$

$$\Rightarrow\; I = \int \frac{du}{u} = \ln(5+x\ln x) + c$$

38) $J = \int \frac{4\ln^3 x + 3}{x(\ln^4 x + 3\ln x)}dx$

$$u = \ln^4 x + 3\ln x \Rightarrow du = \frac{1}{x}(4\ln^3 x + 3)dx$$

$$\Rightarrow J = \int \frac{du}{u} = \ln(\ln^4 x + 3\ln x) + c$$

مشتق توابع زیر را محاسبه کنید.

39)
$$F(t)=\int_{-2}^{\sqrt{t}}\frac{\sin x}{1+\sqrt{1+x^2}}dx$$
$$F'(t)=\frac{1}{2\sqrt{t}}\cdot\frac{\sin\sqrt{t}}{1+\sqrt{1+t}}$$

40)
$$F(x)=\int_{-x}^{x}\frac{dt}{3+t^4}$$
$$F(x)=2\int_{0}^{x}\frac{dt}{3+t^4}\qquad\Rightarrow\qquad F'(x)=\frac{2}{3+t^4}$$

41) ثابت کنید.

الف) $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

حل) اگر قرار دهیم $u=a+b-x$ آنگاه $du=-dx$ و $u(a)=b$, $u(b)=a$ است، پس داریم:

$$\int_a^b f(a+b-x)dx=-\int_b^a f(u)du=\int_a^b f(u)du=\int_a^b f(x)dx$$

ب) $\int_0^t f(x)g(t-x)dx=\int_0^t g(x)f(t-x)dx$

حل) قرار دهید $u=t-x$ پس $du=-dx$ و $u(0)=t$, $u(t)=0$ پس داریم:

$$\int_0^t f(x)g(t-x)dx=-\int_t^0 f(t-u)g(u)du$$
$$=\int_0^t g(u)f(t-u)du=\int_0^t g(x)f(t-x)dx$$

ج) $\int_0^{\frac{\pi}{2}}\sin^m x dx=\int_0^{\frac{\pi}{2}}\cos^m x dx$

حل) اگر قرار دهیم $u=\frac{\pi}{2}-x$ پس $du=-dx$, $u(0)=\frac{\pi}{2}$, $u(\frac{\pi}{2})=0$ لذا داریم:

$$\int_0^{\frac{\pi}{2}}\sin^m x dx=-\int_{\frac{\pi}{2}}^{0}\sin^m(\frac{\pi}{2}-u)du=\int_0^{\frac{\pi}{2}}\cos^m u du=\int_0^{\frac{\pi}{2}}\cos^m x dx$$

42) ثابت کنید. $\int_0^{\pi} f(\sin x)dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)dx$

اگر قرار دهیم $u = \frac{\pi}{2} - x$ آنگاه $u(\pi) = -\frac{\pi}{2}\ ,\ u(0) = \frac{\pi}{2}\ ,\ du = -dx$ پس

$$\int_0^{\pi} f(\sin x)dx = -\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} f(\cos u)du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos u)du = 2\int_0^{\frac{\pi}{2}} f(\cos x)dx$$

علت آخری این است که f(cosu) تابعی زوج است.

43) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

اگـر قـرار دهـیم $u = \frac{\pi}{2} - x$ داریـم $u(\pi) = -\frac{\pi}{2}\ ,\ u(0) = \frac{\pi}{2}\ ,\ du = -dx$ ، لـذا

داریم:

$$\int_0^{\pi} xf(\sin x)dx = -\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (\frac{\pi}{2} - u)f(\sin(\frac{\pi}{2} - u) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2} f(\cos u))du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} uf(\cos u)du = \pi\int_0^{\frac{\pi}{2}} f(\cos x)dx$$

چون تابع $f(\cos a)$ فرد و تابع $f(\cos u)$ زوج است.

44) یک فرمول بازگشتی برای $In=\int \cos^n xdx$ $(n>1)$ پیدا کنید و به کمک آن $\int \cos^4 x\,dx$ را محاسبه کنید.

حل) In را به صورت $In=\int \cos^{n-1} x.\cos xdx$ می نویسیم.

اگر فرض کنیم $dv=\cos x\,dx, u=\cos^{n-1} x$ داریم

$du=-(n-1)\cos^{n-2} x.\sin xdx$ و $V=\sin x$ در نتیجه:

$$In=\sin x.\cos^{n-1} x+(n-1)\int \cos^{n-2} x.\sin^2 x\,dx$$

$$\Rightarrow\ I_n=\sin x.\cos^{n-1} x+(n-1)\ I_{n-2}-(n-1)\ I_n$$

$$\Rightarrow\ nI_n=\sin x.\cos^{n-1} x+(n-1)\ I_{n-2}$$

$$n=2\ \rightarrow\ 2I_2=\sin x.\cos x+x$$

$$n=4\ \rightarrow\ 4I_4=\sin x.\cos^3 x+3I_2$$

$$\Rightarrow I_4=\frac{1}{4}(\sin x.\cos^3 x+\frac{3}{2}(\sin x.\cos x+x))$$

45) انتگرال $\int x^n Lnx\,dx$ را حل کنید.

حل) اگر فرض کنیم. $dv=x^n dx, u=Lnx$ داریم:

$$v=\frac{x^{n+1}}{n+1}\ ,\ du=\frac{dx}{x}$$

پس

$$I_n=\frac{x^{n+1}}{n+1}Lnx-\int \frac{x^n}{n+1}dx=\frac{x^{n+1}}{n+1}Lnx-\frac{x^{n+1}}{(n+1)^2}+c$$

46) فرض کنید $B(m,n)=\int_0^1 x^m(1-x)^n dx$. اولاً تساوی B(m,n)=Bn,m) را ثابت کنید، ثانیاً ثابت کنید که

$$B(m,n)=2\int_0^{\frac{\pi}{2}} \sin^{2m+1} x.\cos^{2n+1} xdx$$

.

حل) اولاً اگر قرار دهیم u=1-x پس $du=-dx$, $u(\circ)=1$, $u(1)=\circ$ پس داریم:

$$B(m,n)=-\int_{\circ}^{1}(1-u)^{m}u^{n}du=\int_{\circ}^{1}x^{n}(1-x)^{m}dx=(n,m)$$

ثانیاً: اگر قرار دهیم $x=\sin^{2}t$ آنگاه $dx=2\sin t\cos dt\,,(1-x)=\cos^{2}t$ و کرانها به $\left[\circ,\frac{\pi}{2}\right]$ تبدیل می شود. پس داریم:

$$B(m,n)=\int_{\circ}^{1}x^{m}(1-x)^{n}dx=\int_{\circ}^{\frac{\pi}{2}}\sin^{2m}t.\cos^{2n}t.2\sin t\cos dt=2\int_{\circ}^{\frac{\pi}{2}}\sin^{2m+1}t.\cos^{2n+1}tdt$$