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Models in Cooperative Game Theory: Crisp, Fuzzy and Multichoice Games

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Preface

This book investigates the classical model of cooperative games with transferable utility (TU-games) and models in which the players have the possibility to cooperate partially, namely fuzzy and multichoice games. In a crisp game the agents are either fully involved or not involved at all in cooperation with some other agents, while in a fuzzy game players are allowed to cooperate with infinitely many different participation levels, varying from non-cooperation to full cooperation. A multichoice game describes an intermediate case in which each player may have a fixed number of activity levels.

Part I of the book is devoted to the most developed model in the theory of cooperative games, that of a classical TU-game with crisp coalitions, which we refer to as crisp game along the book. It presents basic notions, solutions concepts and classes of cooperative crisp games in such a way that allows the reader to use this part as a reference toolbox when studying the corresponding concepts from the theory of fuzzy games (Part II) and from the theory of multichoice games (Part III).

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Part I

Cooperative games with crisp coalitions

Cooperative game theory is concerned primarily with coalitions - groups of players - who coordinate their actions and pool their winnings. Consequentially, one of the problems here is how to divide the extra earnings (or cost savings) among the members of the formed coalition. The basis of this theory was laid by John von Neumann and Oskar Morgenstern in [45] with coalitional games in characteristic function form, known also as transferable utility games (TU-games). Since then several solution concepts for cooperative TU-games have been proposed and several interesting subclasses of TU-games have been introduced. In what follows in this part we present a selection of basic notions, solution concepts and classes of cooperative TU-games that will be extensively used in the next two parts of the book. For recent and more detailed introductory books on the theory of (cooperative) games the reader is referred to [51], [66], where also non-transferable utility games (NTU-games) are treated.

This part of the book is devoted to the most developed model in the theory of cooperative games, that of cooperative games in characteristic function form or cooperative games with transferable utility (TU-games), which we call here cooperative games with crisp coalitions or, simply, crisp games. It is organized as follows. Chapter 1 introduces basic notation, definitions and notions from cooperative game theory dealing with TU-games. In Chapter 2 we consider set solution concepts like the core, the dominance core and stable sets, as well as different core catchers. The relations among these solution concepts are extensively studied. Chapter 3 is devoted to two well known one-point solutions concepts - the Shapley value and the τ-value. We present different formulations of these values, discuss some of their properties and axiomatic characterizations. In Chapter 4 we study three classes of cooperative games with crisp coalitions - totally balanced, convex and clan games. We discuss specific properties of the solution concepts introduced in Chapters 2 and 3 on these classes of games and present specific solution concepts like the concept of a population monotonic allocation scheme for totally balanced games, the constrained egalitarian solution for convex games, and the concept of a bi-monotonic allocation scheme for clan games.

1

Preliminaries

Let N be a non-empty finite set of agents who consider different cooperation possibilities. Each subset $S \subset N$ is referred to as a *crisp coalition*. The set N is called the *grand coalition* and $\emptyset$ is called the *empty coalition*. We denote the collection of coalitions, i.e. the set of all subsets of N by 2^N. For each $S \in 2^N$ we denote by $|S|$ the number of elements of S, and by e^S the characteristic vector of S with $\left(e^S\right)^i = 1$ if $i \in S$, and $\left(e^S\right)^i = 0$ if $i \in N \setminus S$. In the following often $N = \{1, \ldots, n\}$.

Definition 1.1. *A* ***cooperative game*** *in characteristic function form is an ordered pair $\langle N, v\rangle$ consisting of the player set N and the characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$.*

The real number $v(S)$ can be interpreted as the maximal worth or cost savings that the members of S can obtain when they cooperate. Often we identify the game $\langle N, v\rangle$ with its characteristic function v.

A cooperative game in characteristic function form is usually referred to as a transferable utility game (TU-game). A cooperative game might be a non-transferable utility game (NTU-game); the reader is referred to [51] and [66] for an introduction to NTU-games.

Example 1.2. (Glove game) Let $N = \{1, \ldots, n\}$ be divided into two disjoint subsets L and R. Members of L possess a left hand glove, members of R a right hand glove. A single glove is worth nothing, a right-left pair of gloves has value of one euro. This situation can be modeled as a game $\langle N, v\rangle$, where for each $S \in 2^N$ we have $v(S) := \min\{|L \cap S|, |R \cap S|\}$.

The set G^N of characteristic functions of coalitional games with player set N forms with the usual operations of addition and scalar multiplication of functions a $\left(2^{|N|}-1\right)$-dimensional linear space; a basis of this space is supplied by the *unanimity games* u_T, $T \in 2^N \setminus \{\emptyset\}$, that are defined by

$$u_T(S) = \begin{cases} 1 \text{ if } T \subset S, \\ 0 \text{ otherwise.} \end{cases} \tag{1.1}$$

One can easily check that for each $v \in G^N$ we have

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T \text{ with } c_T = \sum_{S : S \subset T} (-1)^{|T|-|S|} v(S). \tag{1.2}$$

The interpretation of the unanimity game u_T is that a gain (or cost savings) of 1 can be obtained if and only if all players in coalition S are involved in cooperation.

Definition 1.3. *A game $v \in G^N$ is called **simple**[1] if $v(S) \in \{0,1\}$ for all $S \in 2^N \setminus \{\emptyset\}$ and $v(\emptyset) = 0$, $v(N) = 1$.*

Note that the unanimity game u_T, $T \in 2^N \setminus \{\emptyset\}$, is a special simple game.

Definition 1.4. *A coalition S is **winning** in the simple game $v \in G^N$ if $v(S) = 1$.*

Definition 1.5. *A coalition S is **minimal winning** in the simple game $v \in G^N$ if $v(S) = 1$ and $v(T) = 0$ for all $T \subset S, T \neq S$.*

Definition 1.6. *A player $i \in N$ is a **dictator** in the simple game $v \in G^N$ if the coalition $\{i\}$ is minimal winning and there are no other minimal winning coalitions.*

Definition 1.7. *Let $v \in G^N$. For each $i \in N$ and for each $S \in 2^N$ with $i \in S$, the **marginal contribution** of player i to the coalition S is $M_i(S,v) := v(S) - v(S \setminus \{i\})$.*

Let $\pi(N)$ be the set of all permutations $\sigma : N \to N$ of N. The set $P^\sigma(i) := \left\{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\right\}$ consists of all predecessors of i with respect to the permutation σ.

Definition 1.8. *Let $v \in G^N$ and $\sigma \in \pi(N)$. The **marginal contribution vector** $m^\sigma(v) \in \mathbb{R}^n$ with respect to σ and v has the i-th coordinate $m_i^\sigma(v) := v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ for each $i \in N$.*

In what follows, we often write m^σ instead of $m^\sigma(v)$ when it is clear which game v we have in mind.

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[1] In some game theory literature a game is simple if it is additionally monotonic (cf. Definition 1.10).

Definition 1.9. *For a game $v \in G^N$ and a coalition $T \in 2^N \setminus \{\emptyset\}$, the* ***subgame*** *with player set T is the game v_T defined by $v_T(S) := v(S)$ for all $S \in 2^T$.*

Hence, v_T is the restriction of v to the set 2^T.

Definition 1.10. *A game $v \in G^N$ is said to be* ***monotonic*** *if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subset T$.*

Definition 1.11. *A game $v \in G^N$ is called* ***non-negative*** *if for each $S \in 2^N$ we have $v(S) \geq 0$.*

Definition 1.12. *A game $v \in G^N$ is* ***additive*** *if $v(S \cup T) = v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.*

An additive game $v \in G^N$ is determined by the vector

$$a = (v(\{1\}), ..., v(\{n\})) \in \mathbb{R}^n \tag{1.3}$$

since $v(S) = \sum_{i \in S} a_i$ for all $S \in 2^N$. Additive games form an n-dimensional linear subspace of G^N. A game $v \in G^N$ is called *inessential* if it is an additive game. For an inessential game there is no problem how to divide $v(N)$ because $v(N) = \sum_{i \in N} v(\{i\})$ (and also $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subset N$)[2].

Most of the cooperative games arising from real life situations are superadditive games.

Definition 1.13. *A game $v \in G^N$ is* ***superadditive*** *if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.*

Of course, in a superadditive game we have $v\left(\cup_{i=1}^k S_i\right) \geq \sum_{i=1}^k v(S_i)$ if $S_1, \ldots, S_k$ are pairwise disjoint coalitions. Especially $v(N) \geq \sum_{i=1}^k v(S_i)$ for each partition $(S_1, \ldots, S_k)$ of N; in particular $v(N) \geq \sum_{i=1}^n v(i)$. Note that the game in Example 1.2 is superadditive. In a superadditive game it is advantageous for the players to cooperate. The set of (characteristic functions of) superadditive games form a *cone* in G^N, i.e. for all v and w that are superadditive we have that $\alpha v + \beta w$ is also a superadditive game, where $\alpha, \beta \in \mathbb{R}_+$.

Definition 1.14. *A game $v \in G^N$ for which $v(N) > \sum_{i=1}^n v(i)$ is said to be an N-**essential game** .*

In what follows in Part I, the notion of a *balanced game* will play an important role.

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[2] Given a game $v \in G^N$ and a coalition $\{i, \ldots, k\} \subset N$, we will often write $v(i, \ldots, k)$ instead of $v(\{i, \ldots, k\})$.

Definition 1.15. *A map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ is called a* ***balanced map*** *if $\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)\, e^S = e^N$.*

Definition 1.16. *A collection B of coalitions is called* ***balanced*** *if there is a balanced map λ such that $B = \left\{S \in 2^N \mid \lambda(S) > 0\right\}$.*

Definition 1.17. *A game $v \in G^N$ is* ***balanced*** *if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have*

$$\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)\, v(S) \leq v(N). \tag{1.4}$$

Let us consider now two games $v, w \in G^N$ and answer the question "When can we say that v and w are 'essentially' the same?"

Definition 1.18. *Let $v, w \in G^N$. The game w is* ***strategically equivalent*** *to the game v if there exist $k > 0$ and an additive game a (cf. (1.3)) such that $w(S) = kv(S) + \sum_{i\in S} a_i$ for all $S \in 2^N \setminus \{\emptyset\}$.*

One may think that w arises out of v by the following changes:

- the unit of payoffs is changed, where the exchange rate is k;
- in the game w each player is given either a bonus (if $a_i > 0$) or a fee (if $a_i < 0$) before the distribution of $kv(N)$ among the players starts.

Notice that the strategic equivalence is an equivalence relation on the set G^N, i.e. we have:

- (Reflexivity) The game v is strategically equivalent to itself (take $k = 1$ and $a_i = 0$ for each $i \in N$);
- (Symmetry) If w is strategically equivalent to v, then v is strategically equivalent to w (if for all coalitions $S \subset N$, $w(S) = kv(S) + \sum_{i\in S} a_i$, then $v(S) = \frac{1}{k} w(S) - \sum_{i\in S} \frac{a_i}{k}$ and $\frac{1}{k} > 0$);
- (Transitivity) If w is strategically equivalent to v and u is strategically equivalent to w, then u is strategically equivalent to v ($w(S) = kv(S) + a(S)$ and $u(S) = lw(S) + b(S)$ imply $u(S) = lkv(S) + (la(S) + b(S))$, where $a(S) := \sum_{i\in S} a_i$).

For most solution concepts – as we will see later – it is sufficient to look only at one of the games in an (strategic) equivalence class. One considers often games in an equivalence class that are in (α, β)-form for $\alpha, \beta \in \mathbb{R}$.

Definition 1.19. *Let $\alpha, \beta \in \mathbb{R}$. A game $v \in G^N$ is called a* ***game in** (α, β)**-form*** *if $v(i) = \alpha$ for all $i \in N$ and $v(N) = \beta$.*

Theorem 1.20. *Each N-essential game $v \in G^N$ is strategically equivalent to a game $w \in G^N$ in $(0, 1)$-form. This game is unique.*

Proof. For some $k > 0$ and $a_1, ..., a_n \in \mathbb{R}$ we try to find a game w with $w(S) = kv(S) + a(S)$ for all $S \in 2^N \setminus \{\emptyset\}$, $w(\{i\}) = 0$ for all $i \in N$, and $w(N) = 1$. Then necessarily

$$w(i) = 0 = kv(i) + a_i, \tag{1.5}$$

$$w(N) = 1 = kv(N) + \sum_{i \in N} a_i. \tag{1.6}$$

Then $w(N) - \sum_{i \in N} w(i) = 1 = k\left(v(N) - \sum_{i \in N} v(i)\right)$ by (1.5) and (1.6). Hence, $k = \frac{1}{v(N) - \sum_{i \in N} v(i)}$. From (1.5) we derive $a_i = -\frac{v(i)}{v(N) - \sum_{i \in N} v(i)}$. If

we take for all $S \in 2^N \setminus \{\emptyset\}$, $w(S) = \frac{v(S) - \sum_{i \in S} v(i)}{v(N) - \sum_{i \in N} v(i)}$, then we obtain the unique game w in $(0,1)$-form, which is strategically equivalent to v.

Definition 1.21. *A game $v \in G^N$ is called **zero-normalized** if for all $i \in N$ we have $v(i) = 0$.*

One can easily check that each game $v \in G^N$ is strategically equivalent to a zero-normalized game $w \in G^N$, where $w(S) = v(S) - \sum_{i \in S} v(i)$.

Definition 1.22. *A game $v \in G^N$ is said to be **zero-monotonic** if its zero-normalization is monotonic.*

It holds that a game which is strategically equivalent to a zero-monotonic game is also zero-monotonic.

We turn now to one of the basic questions in the theory of cooperative TU-games: "If the grand coalition forms, how to divide the profit or cost savings $v(N)$?"

This question is approached with the aid of solution concepts in cooperative game theory like cores, stable sets, bargaining sets, the Shapley value, the τ-value, the nucleolus. A solution concept gives an answer to the question of how the reward (cost savings) obtained when all players in N cooperate should be distributed among the individual players while taking account of the potential reward (cost savings) of all different coalitions of players. Hence, a solution concept assigns to a coalitional game at least one payoff vector $x = (x_i)_{i \in N} \in \mathbb{R}^n$, where x_i is the payoff allocated to player $i \in N$. A selection of (set-valued and one-point) solution concepts which will be used along this book, their axiomatic characterizations and interrelations will be given in Chapters 2-4.

Definition 1.23. *A **set-valued solution** (or a multisolution) is a multifunction $F : G^N \to\to \mathbb{R}^n$.*

Definition 1.24. *An **one-point solution** (or a single-valued rule) is a map $f : G^N \to \mathbb{R}^n$.*

We mention now some desirable properties for one-point solution concepts. Extensions of these properties to set-valued solution concepts are straightforward.

Definition 1.25. *Let $f : G^N \to \mathbb{R}^n$. Then f satisfies*

(i) ***individual rationality*** *if $f_i(v) \geq v(i)$ for all $v \in G^N$ and $i \in N$.*
(ii) ***efficiency*** *if $\sum_{i=1}^{n} f_i(v) = v(N)$ for all $v \in G^N$.*
(iii) ***relative invariance with respect to strategic equivalence*** *if for all $v, w \in G^N$, all additive games $a \in G^N$, and all $k > 0$ we have that $w = kv + a$ implies $f(kv + a) = kf(v) + a$.*
(iv) the ***dummy player property*** *if $f_i(v) = v(i)$ for all $v \in G^N$ and for all dummy players i in v, i.e. players $i \in N$ such that $v(S \cup \{i\}) = v(S) + v(i)$ for all $S \in 2^{N \setminus \{i\}}$.*
(v) the ***anonymity property*** *if $f(v^\sigma) = \sigma^*(f(v))$ for all $\sigma \in \pi(N)$. Here v^σ is the game with $v^\sigma(\sigma(U)) := v(U)$ for all $U \in 2^N$ or $v^\sigma(S) = v\left(\sigma^{-1}(S)\right)$ for all $S \in 2^N$ and $\sigma^* : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $(\sigma^*(x))_{\sigma(k)} := x_k$ for all $x \in \mathbb{R}^n$ and $k \in N$.*
(vi) ***additivity*** *if $f(v + w) = f(v) + f(w)$ for all $v, w \in G^N$.*

We end this chapter by recalling some definitions and results from linear algebra which are used later.

Definition 1.26. *Let V and W be vector spaces over $\mathbb{R}$. Let $L : V \to W$ be a map. Then L is called a* ***linear transformation*** *(linear map, linear operator) from V into W if for all $x, y \in V$ and all $\alpha, \beta \in \mathbb{R}$ we have $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.*

Definition 1.27. *A set W is a (linear)* ***subspace*** *of the vector space V if $W \subset V$, $0 \in W$, and W is closed with respect to addition and scalar multiplication, i.e. for all $x, y \in W$ we have $x + y \in W$, and for each $x \in W$ and $\alpha \in \mathbb{R}$, also $\alpha x \in W$ holds.*

Definition 1.28. *A subset C of a vector space V over $\mathbb{R}$ is called* ***convex*** *if for all $x, y \in C$ and all $\alpha \in (0, 1)$ we have $\alpha x + (1 - \alpha) y \in C$.*

A geometric interpretation of a convex set is that with each pair x, y of points in it, the line segment with x, y as endpoints is also in the set.

Definition 1.29. *Let C be a convex set. A point $x \in C$ is called an* ***extreme point*** *of C if there do not exist $x_1, x_2 \in C$ with $x_1 \neq x$, $x_2 \neq x$ and $\alpha \in (0, 1)$ such that $x = \alpha x_1 + (1 - \alpha) x_2$. The set of extreme points of a convex set C will be denoted by $ext(C)$.*

Definition 1.30. *A set H of points in $\mathbb{R}^n$ is called a* ***hyperplane*** *if it is the set of solutions of a linear equation $a_1 x_1 + ... + a_n x_n = b$, with $(a_1, ..., a_n) \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. A hyperplane separates a (linear) space in two (linear) halfspaces. Let A be an $n \times p$ matrix and $b \in \mathbb{R}^p$; a set $P = \left\{x \in \mathbb{R}^n \mid x^T A \geq b^T\right\}$ is called a* ***polyhedral set****.*

The following theorem gives a characterization of extreme points of a polyhedral set.

Theorem 1.31. *Let A be an $n \times p$ matrix, $b \in \mathbb{R}^p$ and let P be the polyhedral set of solutions of the set of inequalities $x^T A \geq b^T$. For $x \in \mathbb{R}$ let tight(x) be the set of columns $\left\{Ae^j \mid x^T Ae^j = b_j\right\}$ of A where the corresponding inequalities are equalities for x, and where for each $j \in N$, e^j is the j-th standard basis vector in $\mathbb{R}^n$. Then x is an extreme point of P iff tight(x) is a complete system of vectors in $\mathbb{R}^n$.*

The next theorem is known as the duality theorem from linear programming theory.

Theorem 1.32. *Let A be an $n \times p$ matrix, $b \in \mathbb{R}^p$ and $c \in \mathbb{R}^n$. Then* $\min\left\{x^T c \mid x^T A \geq b^T\right\} = \max\left\{b^T y \mid Ay = c, y \geq 0\right\}$ *if* $\left\{x \in \mathbb{R}^n \mid x^T A \geq b^T\right\} \neq \emptyset$ *and* $\{y \in \mathbb{R}^p \mid Ay = c, y \geq 0\} \neq \emptyset$.

Definition 1.33. *Let V be a vector space and $A \subset V$. The* ***convex hull*** *co(A) of A is the set*

$$\left\{x \in V \mid \exists p \in \mathbb{N}, \alpha \in \Delta^p, v_1, ..., v_p \in A \text{ s.t. } \sum_{i=1}^{p} \alpha_i v_i = x\right\},$$

where $\Delta^p = \left\{q \in \mathbb{R}^p_+ \mid \sum_{i=1}^p q_i = 1\right\}$ is the $(p-1)$-dimensional unit simplex.

2

Cores and related solution concepts

In this chapter we consider payoff vectors $x = (x_i)_{i \in N} \in \mathbb{R}^n$, with x_i being the payoff to be given to player $i \in N$, under the condition that cooperation in the grand coalition is reached. Clearly, the actual formation of the grand coalition is based on the agreement of all players upon a proposed payoff in the game. Such an agreement is, or should be, based on all other cooperation possibilities for the players and their corresponding payoffs.

2.1 Imputations, cores and stable sets

We note first that only payoff vectors $x \in \mathbb{R}^n$ satisfying $\sum_{i \in N} x_i \leq v(N)$ are reachable in the game $v \in G^N$ and the set of such payoff vectors is nonempty and convex. More precisely, it is a halfspace of $\mathbb{R}^n$. We denote this set by $I^{**}(v)$, i.e.

$$I^{**}(v) := \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i \leq v(N) \right\}.$$

However, to have any chance of being agreed upon, a payoff vector should satisfy *efficiency*, i.e.

$$\sum_{i \in N} x_i = v(N).$$

To motivate the efficiency condition we argue that $\sum_{i \in N} x_i \geq v(N)$ should also hold.

Suppose that $\sum_{i\in N} x_i < v(N)$. In this case we would have

$$a = v(N) - \sum_{i\in N} x_i > 0.$$

Then the players can still form the grand coalition and receive the better payoff $y = (y_1, \ldots, y_n)$ with $y_i = x_i + \frac{a}{n}$ for all $i \in N$.

We denote by $I^*(v)$ the set of efficient payoff vectors in the coalitional game $v \in G^N$, i.e.

$$I^*(v) := \left\{ x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(N) \right\}$$

and, clearly, $I^*(v) \neq \emptyset$. This convex set is referred to as the *preimputation set* of the game $v \in G^N$. It is a hyperplane in $\mathbb{R}^n$. Clearly, $I^*(v) \subset I^{**}(v)$.

Now, note that if the proposed allocation $x \in I^*(v)$ is such that there is at least one player $i \in N$ whose payoff x_i satisfies $x_i < v(i)$, the grand coalition would never form. The reason is that such a player would prefer not to cooperate since acting on his own he can obtain more.

Hence, the *individual rationality* condition

$$x_i \geq v(i) \text{ for all } i \in N$$

should hold in order that a payoff vector has a real chance to be realized in the game.

Definition 2.1. *A payoff vector $x \in \mathbb{R}^n$ is an **imputation** for the game $v \in G^N$ if it is efficient and individually rational, i.e.*
(i) $\sum_{i\in N} x_i = v(N)$;
(ii) $x_i \geq v(i)$ for all $i \in N$.

We denote by $I(v)$ the set of imputations of $v \in G^N$. Clearly, $I(v)$ is empty if and only if $v(N) < \sum_{i\in N} v(i)$. Further, for an additive game (cf. Definition 1.12),

$$I(v) = \{(v(1), \ldots, v(n))\}.$$

The next theorem shows that N-essential games (cf. Definition 1.14) always have infinitely many imputations. Moreover, $I(v)$ is a simplex with extreme points $f^1, \ldots, f^n$, where for each $i \in N$, $f^i = \left(f_1^i, \ldots, f_j^i, \ldots, f_n^i\right)$ with

$$f_j^i = \begin{cases} v(i) & \text{if } i \neq j, \\ v(N) - \sum_{k\in N\setminus\{i\}} v(k) & \text{if } i = j. \end{cases} \tag{2.1}$$

Theorem 2.2. *Let $v \in G^N$. If v is N-essential, then*
(i) $I(v)$ is an infinite set.
(ii) $I(v)$ is the convex hull of the points $f^1, \ldots, f^n$.

Proof. (i) Since $v \in G^N$ is an N-essential game we have $a = v(N) - \sum_{i\in N} x_i > 0$. For any n-tuple $b = (b_1, \ldots, b_n)$ of nonnegative numbers such that $\sum_{i\in N} b_i = a$, the payoff vector $x' = (x'_1, \ldots, x'_n)$ with $x'_i = v(i) + b_i$ for all $i \in N$ is an imputation.

(ii) This follows from Theorem 1.31 by noting that

$$I(v) = \left\{x \in \mathbb{R}^n \mid x^T A \geq b^T\right\},$$

where A is the $n \times (n+2)$-matrix with columns $e^1, \ldots, e^n, 1^n, -1^n$ and

$$b = (v(1), \ldots, v(n), v(N), -v(N)),$$

where for each $i \in N$, e^i is the i-th standard basis in $\mathbb{R}^n$ and 1^n is the vector in $\mathbb{R}^n$ with all coordinates equal to 1.

Since the imputation set of an N-essential game is too large according to the above theorem, there is a need for some criteria to single out those imputations that are most likely to occur. In this way one obtains subsets of $I(v)$ as *solution concepts*.

The first (set-valued) solution concept we would like to study is the *core* of a game (cf. [30]).

Definition 2.3. *The* ***core*** $C(v)$ *of a game* $v \in G^N$ *is the set*

$$\left\{x \in I(v) \mid \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\right\}.$$

If $x \in C(v)$, then no coalition S has an incentive to split off if x is the proposed reward allocation in N, because the total amount $\sum_{i\in S} x_i$ allocated to S is not smaller than the amount $v(S)$ which the players can obtain by forming the subcoalition. If $C(v) \neq \emptyset$, then elements of $C(v)$ can easily be obtained because the core is defined with the aid of a finite system of linear inequalities. The core is a polytope.

In [9] and [59] one can find a characterization of games with a nonempty core that we present in the next theorem.

Theorem 2.4. *Let* $v \in G^N$. *Then the following two assertions are equivalent:*
(i) $C(v) \neq \emptyset$,
(ii) The game v *is balanced (cf. Definition 1.17).*

Proof. First we note that $C(v) \neq \emptyset$ iff

$$v(N) = \min\left\{\sum_{i\in N} x_i \mid \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\right\}. \tag{2.2}$$

By Theorem 1.32, equality (2.2) holds iff

$$v(N) = \max\left\{ \sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)v(S) \mid \sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)e^S = e^N, \lambda \geq 0 \right\} \quad (2.3)$$

(take for A the matrix with the characteristic vectors e^S as columns). Now (2.3) holds iff (1.4) holds. Hence, (i) and (ii) are equivalent.

Remark 2.5. The core is relative invariant with respect to strategic equivalence (cf. Definition 1.25(iii)): if $w \in G^N$ is strategically equivalent to $v \in G^N$, say $w = kv + a$, then

$$C(w) = kC(v) + a\,(:= \{x \in \mathbb{R}^n \mid x = ky + a \text{ for some } y \in C(v)\})\,.$$

Other subsets of imputations which are solution concepts for coalitional games are the *dominance core* (D-core) and *stable sets* (cf. [45]). They are defined based on the following dominance relation over vectors in $\mathbb{R}^n$.

Definition 2.6. *Let $v \in G^N$, $x, y \in I(v)$, and $S \in 2^N \setminus \{\emptyset\}$. We say that x* ***dominates*** *y* ***via coalition*** *S, and denote it by $x \operatorname{dom}_S y$ if*
(i) $x_i > y_i$ for all $i \in S$,
(ii) $\sum_{i\in S} x_i \leq v(S)$.

Note that if (i) holds, then the payoff x is better than the payoff y for all members of S; condition (ii) guarantees that the payoff x is reachable for S.

Definition 2.7. *Let $v \in G^N$ and $x, y \in I(v)$. We say that x* ***dominates*** *y, and denote it by $x \operatorname{dom} y$ if there exists $S \in 2^N \setminus \{\emptyset\}$ such that $x \operatorname{dom}_S y$.*

Proposition 2.8. *Let $v \in G^N$ and $S \in 2^N \setminus \{\emptyset\}$. Then the relations* dom_S *and* dom *are irreflexive. Moreover,* dom_S *is transitive and antisymmetric.*

Proof. That dom_S and dom are irreflexive follows from the fact that for $x \in I(v)$ there is no $S \in 2^N \setminus \{\emptyset\}$ such that $x_i > x_i$ for all $i \in S$.

To prove that dom_S is transitive take $x, y, z \in I(v)$ such that $x \operatorname{dom}_S y$ and $y \operatorname{dom}_S z$. Then $x_i > z_i$ for all $i \in S$. So $x \operatorname{dom}_S z$.

To prove that dom_S is antisymmetric, suppose $x \operatorname{dom}_S y$. Then $x_i > y_i$ for all $i \in S$, i.e. there is no $i \in S$ such that $y_i > x_i$. Hence, $y \operatorname{dom}_S x$ does not hold.

For $S \in 2^N \setminus \{\emptyset\}$ we denote by $D(S)$ the set of imputations which are dominated via S; note that players in S can successfully protest against any imputation in $D(S)$.

Definition 2.9. *The* ***dominance core*** *(D-**core**) $DC(v)$ of a game $v \in G^N$ consists of all undominated elements in $I(v)$, i.e. the set $I(v) \setminus \cup_{S\in 2^N\setminus\{\emptyset\}} D(S)$.*

It turns out that $DC(v)$ is also a convex set; moreover, it is a polytope and relative invariant with respect to strategic equivalence.

For $v \in G^N$ and $A \subset I(v)$ we denote by $\mathrm{dom}\,(A)$ the set consisting of all imputations that are dominated by some element in A. Note that $DC(v) = I(v) \setminus \mathrm{dom}(I(v))$.

Definition 2.10. *For $v \in G^N$ a subset K of $I(v)$ is called a **stable set** if the following conditions hold:*
(i) (Internal stability) $K \cap \mathrm{dom}(K) = \emptyset$,
(ii) (External stability) $I(v) \setminus K \subset \mathrm{dom}(K)$.

This notion was introduced by von Neumann and Morgenstern (cf. [45]) with the interpretation that a stable set corresponds to a "standard of behavior", which, if generally accepted, is self-enforcing.

The two conditions in Definition 2.10 can be interpreted as follows:

- By external stability, an imputation outside a stable set K seems unlikely to become established: there is always a coalition that prefers one of the achievable imputations inside K, implying that there would exist a tendency to shift to an imputation in K;
- By internal stability, all imputations in K are "equal" with respect to the dominance relation via coalitions, i.e. there is no imputation in K that is dominated by another imputation in K.

Note that for a game $v \in G^N$ the set K is a stable set if and only if K and $\mathrm{dom}(K)$ form a partition of $I(v)$. In principle, a game may have many stable sets or no stable set.

Theorem 2.11. *Let $v \in G^N$ and K be a stable set of v. Then*
(i) $C(v) \subset DC(v) \subset K$;
(ii) If v is superadditive, then $DC(v) = C(v)$;
(iii) If $DC(v)$ is a stable set, then there is no other stable set.

Proof. (i) In order to show that $C(v) \subset DC(v)$, let us suppose that there is $x \in C(v)$ such that $x \notin DC(v)$. Then there is an $y \in I(v)$ and a coalition $S \in 2^N \setminus \{\emptyset\}$ such that $y\,\mathrm{dom}_S\,x$. Then $v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i$ which implies that $x \notin C(v)$.

To prove next that $DC(v) \subset K$ it is sufficient to show that $I(v) \setminus K \subset I(v) \setminus DC(v)$. Take $x \in I(v) \setminus K$. By the external stability of K there is a $y \in K$ with $y\ \mathrm{dom}\,x$. The elements in $DC(v)$ are not dominated. So $x \notin DC(v)$, i.e. $x \in I(v) \setminus DC(v)$.

(ii) We divide the proof of this assertion into two parts.

(ii.1) We show that for an $x \in I(v)$ with $\sum_{i \in S} x_i < v(S)$ for some $S \in 2^N \setminus \{\emptyset\}$, there is $y \in I(v)$ such that $y\,\mathrm{dom}_S\,x$. Define y as follows. If $i \in S$, then $y_i := x_i + \frac{v(S) - \sum_{i \in S} x_i}{|S|}$. If $i \notin S$, then $y_i := v\,(i) + \frac{v(N) - v(S) - \sum_{i \in N \setminus S} v(i)}{|N \setminus S|}$. Then $y \in I(v)$, where for the proof of $y_i \geq v\,(i)$ for $i \in N \setminus S$ we use the superadditivity of the game. Furthermore, $y\,\mathrm{dom}_S\,x$.

(ii.2) In order to show $DC(v) = C(v)$ we have, in view of (i), only to prove that $DC(v) \subset C(v)$. Suppose $x \in DC(v)$. Then there is no $y \in I(v)$ such that $y \operatorname{dom} x$. In view of (ii.1) we then have $\sum_{i \in S} x_i \geq v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Hence, $x \in C(v)$.

(iii) Suppose $DC(v)$ is a stable set. Let K also be stable. By (i) we have $DC(v) \subset K$. To prove $K = DC(v)$, we have to show that $K \setminus DC(v) = \emptyset$. Suppose, to the contrary, that there is $x \in K \setminus DC(v)$. By the external stability of $DC(v)$ there is $y \in DC(v)\,(\subset K)$ such that $y \operatorname{dom} x$. This is a contradiction to the internal stability of K. Hence $K \setminus DC(v) = \emptyset$ holds.

In addition to the relations among the core, the dominance core and the stable sets as established in Theorem 2.11, we state next without proof some additional results that will be used in the next parts of the book.

Theorem 2.12. *Let $v \in G^N$. Then*
(i) If $DC(v) \neq \emptyset$ and $v(N) \geq v(S) + \sum_{i \in N \setminus S} v(i)$ for each $S \subset N$, then $C(v) = DC(v)$.
(ii) If $C(v) \neq DC(v)$, then $C(v) = \emptyset$.

For details with respect to these relations the reader is referred to [23], [53], [61], and [66].

Another core-like solution concept which is related to the norm of equity is the equal division core introduced in [56]. Given a game $v \in G^N$, the *equal division core* $EDC(v)$ is the set

$$\left\{ x \in I(v) \mid \nexists S \in 2^N \setminus \{\emptyset\} \text{ s.t. } \frac{v(S)}{|S|} > x_i \text{ for all } i \in S \right\},$$

consisting of efficient payoff vectors for the grand coalition which cannot be improved upon by the equal division allocation of any subcoalition. It is clear that the core of a cooperative game is included in the equal division core of that game. The reader can find axiomatic characterizations of this solution concept on two classes of cooperative games in [7].

2.2 The core cover, the reasonable set and the Weber set

In this section we introduce three sets related to the core, namely the *core cover* (cf. [68]), the *reasonable set* (cf. [29], [40], and [42]), and the *Weber set* (cf. [74]). All these sets can be seen as "core catchers" in the sense that they all contain the core of the corresponding game as a subset.

In the definition of the core cover the *upper vector* $M(N, v)$ and the *lower vector* $m(v)$ of a game $v \in G^N$ play a role.

For each $i \in N$, the i-th coordinate $M_i(N, v)$ of the upper vector $M(N, v)$ is the marginal contribution of player i to the grand coalition (cf. Definition

1.7); it is also called the *utopia payoff* for player i in the grand coalition in the sense that if player i wants more, then it is advantageous for the other players in N to throw player i out.

Definition 2.13. *Let $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$. The* ***remainder*** *$R(S,i)$ of player i in the coalition S is the amount which remains for player i if coalition S forms and all other players in S obtain their utopia payoffs, i.e.*

$$R(S,i) := v(S) - \sum_{j \in S \setminus \{i\}} M_j(N,v).$$

For each $i \in N$, the i-th coordinate $m_i(v)$ of the *lower vector* $m(v)$ is then defined by

$$m_i(v) := \max_{S : i \in S} R(S,i).$$

We refer to $m_i(v)$ also as the *minimum right payoff* for player i, since this player has a reason to ask at least $m_i(v)$ in the grand coalition N, by arguing that he can obtain that amount also by drumming up a coalition S with $m_i(v) = R(S,i)$ and making all other players in S happy with their utopia payoffs.

Definition 2.14. *The* ***core cover*** *$CC(v)$ of $v \in G^N$ consists of all imputations which are between $m(v)$ and $M(N,v)$ (in the usual partial order of $\mathbb{R}^n$), i.e.*

$$CC(v) := \{x \in I(v) \mid m(v) \le x \le M(N,v)\}.$$

That $CC(v)$ is a core catcher follows from the following theorem, which tells us that the lower (upper) vector is a lower (upper) bound for the core.

Theorem 2.15. *Let $v \in G^N$ and $x \in C(v)$. Then $m(v) \le x \le M(N,v)$ i.e. $m_i(v) \le x_i \le M_i(N,v)$ for all $i \in N$.*

Proof. (i) $x_i = x(N) - x(N \setminus \{i\}) = v(N) - x(N \setminus \{i\}) \le v(N) - v(N \setminus \{i\}) = M_i(N,v)$ for each $i \in N$.

(ii) In view of (i), for each $S \subset N$ and each $i \in S$ we have

$$x_i = x(S) - x(S \setminus \{i\}) \ge v(S) - \sum_{j \in S \setminus \{i\}} M_j(N,v) = R(S,i).$$

So, $x_i \ge \max_{S : i \in S} R(S,i) = m_i(v)$ for each $i \in S$.

Another core catcher for a game $v \in G^N$ is introduced (cf. [42]) as follows.

Definition 2.16. *The* ***reasonable set*** *$R(v)$ of a game $v \in G^N$ is the set*

$$\left\{ x \in \mathbb{R}^n \mid v(i) \le x_i \le \max_{S : i \in S} \left(v(S) - v(S \setminus \{i\})\right) \right\}.$$

Obviously, $C(v) \subset CC(v) \subset R(v)$.

The last core catcher for a game $v \in G^N$ we introduce (cf. [74]) is the Weber set. In its definition the marginal contribution vectors (cf. Definition 1.8) play a role.

Definition 2.17. *The* ***Weber set*** $W(v)$ *of a game* $v \in G^N$ *is the convex hull of the* $n!$ *marginal vectors* $m^\sigma(v)$*, corresponding to the* $n!$ *permutations* $\sigma \in \pi(N)$.

Here $m^\sigma(v)$ is the vector with

$$\begin{aligned} m^\sigma_{\sigma(1)}(v) &:= v(\sigma(1)), \\ m^\sigma_{\sigma(2)}(v) &:= v(\sigma(1), \sigma(2)) - v(\sigma(1)), \\ &\vdots \\ m^\sigma_{\sigma(k)}(v) &:= v(\sigma(1), \ldots, \sigma(k)) - v(\sigma(1), \ldots, \sigma(k-1)) \end{aligned}$$

for each $k \in N$. The payoff vector m^σ can be created as follows. Let the players enter a room one by one in the order $\sigma(1), \ldots, \sigma(n)$ and give each player the marginal contribution he creates by entering.

The Weber set is a core catcher as shown in

Theorem 2.18. *Let* $v \in G^N$. *Then* $C(v) \subset W(v)$.

Proof. If $|N| = 1$, then $I(v) = C(v) = W(v) = \{(v(1))\}$.

- For $|N| = 2$ we consider two cases: $I(v) = \emptyset$ and $I(v) \neq \emptyset$. If $I(v) = \emptyset$, then $C(v) \subset I(v) = \emptyset \subset W(v)$. If $I(v) \neq \emptyset$, then we let

  $$x' = (v(1), v(1,2) - v(1))$$

  and

  $$x'' = (v(2), v(1,2) - v(2)),$$

  and note that

  $$\begin{aligned} C(v) = I(v) &= co\,\{x', x''\} \\ &= co\,\{m^\sigma(v) \mid \sigma : \{1,2\} \to \{1,2\}\} = W(v). \end{aligned}$$

- We proceed by induction on the number of players. So, suppose $|N| = n > 2$ and suppose that the core is a subset of the Weber set for all games with number of players smaller than n.
- Since $C(v)$ and $W(v)$ are convex sets we need only to show that $x \in ext\,(C(v))$ implies $x \in W(v)$. Take $x \in ext(C(v))$. Then it follows from Theorem 1.31 that there exists $T \in 2^N \setminus \{\emptyset, N\}$ with $x(T) = v(T)$. Consider the $|T|$-person game u and the $(n - |T|)$-person game w defined by

$$u(S) = v(S) \text{ for each } S \in 2^T,$$
$$w(S) = v(T \cup S) - v(T) \text{ for each } S \in 2^{N\setminus T}.$$

Then, obviously, $x^T \in C(u)$, and also $x^{N\setminus T} \in C(w)$ because $x^{N\setminus T}(S) = x(S) = x(T \cup S) - x(T) \geq v(T \cup S) - x(T) = v(T \cup S) - v(T) = w(S)$ for all $S \in 2^{N\setminus T}$ and

$$\sum_{i \in N\setminus T} x_i^{N\setminus T} = x(N) - x(T) = v(N) - v(T) = w(N \setminus T).$$

Since $|T| < n$, $|N \setminus T| < n$, the induction hypothesis implies that $x^T \in W(u)$ and $x^{N\setminus T} \in W(w)$.
Then $x = x^T \times x^{N\setminus T} \in W(u) \times W(w) \subset W(v)$. This last inclusion can be seen as follows. The extreme points of $W(u) \times W(w)$ are of the form (m^ρ, m^τ), where $\rho : \{1, 2, \ldots, |T|\} \to T$ and $\tau : \{1, 2, \ldots, |N \setminus T|\} \to N \setminus T$ are bijections and $m^\rho \in \mathbb{R}^{|T|}, m^\tau \in \mathbb{R}^{|N\setminus T|}$ are given by

$$\begin{aligned}
m^\rho_{\rho(1)} &:= u(\rho(1)) = v(\rho(1)), \\
m^\rho_{\rho(2)} &:= u(\rho(1), \rho(2)) - u(\rho(1)) = v(\rho(1), \rho(2)) - v(\rho(1)), \\
&\vdots \\
m^\rho_{\rho(|T|)} &:= u(T) - u(T \setminus \{\rho(|T|)\}) = v(T) - v(T \setminus \{\rho(|T|)\}), \\
m^\tau_{\tau(1)} &:= w(\tau(1)) = v(T \cup \{\tau(1)\}) - v(T), \\
m^\tau_{\tau(2)} &:= w(\tau(1), \tau(2)) - w(\tau(1)) \\
&\qquad = v(T \cup \{\tau(1), \tau(2)\}) - v(T \cup \{\tau(1)\}), \\
&\vdots \\
m^\tau_{\tau(|N\setminus T|)} &:= w(N \setminus T) - w((N \setminus T) \setminus \{\tau(|N \setminus T|)\}) \\
&\qquad = v(N) - v(N \setminus \{\tau(|N \setminus T|)\}).
\end{aligned}$$

Hence, $(m^\rho, m^\tau) \in \mathbb{R}^n$ corresponds to the marginal vector m^σ in $W(v)$ where $\sigma : N \to N$ is defined by

$$\sigma(i) := \begin{cases} \rho(i) & \text{if } 1 \leq i \leq |T|, \\ \tau(i - |T|) & \text{if } |T| + 1 \leq i \leq n. \end{cases}$$

So, we have proved that $ext(W(u) \times W(w)) \subset W(v)$. Since $W(v)$ is convex, $W(u) \times W(w) \subset W(v)$. We have proved that $x \in ext(C(v))$ implies $x \in W(v)$.

For another proof of Theorem 2.18 the reader is referred to [22].

3

The Shapley value and the τ-value

The Shapley value and the τ-value are two interesting one-point solution concepts in cooperative game theory. In this chapter we discuss different formulations of these values, some of their properties and give axiomatic characterizations of the Shapley value.

3.1 The Shapley value

The Shapley value (cf. [58]) associates to each game $v \in G^N$ one payoff vector in $\mathbb{R}^n$. For a very extensive and interesting discussion on this value the reader is referred to [54].

The first formulation of the Shapley value uses the marginal vectors (see Definition 1.8) of a cooperative TU-game.

Definition 3.1. *The* ***Shapley value*** $\Phi(v)$ *of a game* $v \in G^N$ *is the average of the marginal vectors of the game, i.e.*

$$\Phi(v) := \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^{\sigma}(v). \tag{3.1}$$

With the aid of (3.1) one can provide a probabilistic interpretation of the Shapley value as follows. Suppose we draw from an urn, containing the elements of $\pi(N)$, a permutation σ (with probability $\frac{1}{n!}$). Then we let the players enter a room one by one in the order σ and give each player the marginal contribution created by him. Then, for each $i \in N$, the i-th coordinate $\Phi_i(v)$ of $\Phi(v)$ is the expected payoff of player i according to this random procedure.

By using Definition 1.8 one can rewrite (3.1) obtaining

$$\Phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} \left(v \left(P^{\,\sigma}(i) \cup \{i\} \right) - v \left(P^{\,\sigma}(i) \right) \right). \tag{3.2}$$

Example 3.2. Let $N = \{1, 2, 3\}$, $v(1, 2) = -2$, $v(S) = 0$ if $S \neq \{1, 2\}$. Then the Shapley value is the average of the vectors $(0, -2, 2)$, $(0, 0, 0)$, $(-2, 0, 2)$, $(0, 0, 0)$, $(0, 0, 0)$, and $(0, 0, 0)$, i.e.

$$\Phi(v) = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right).$$

Remark 3.3. The game in Example 3.2 shows that the Shapley value needs not to be individually rational (cf. Definition 1.25(i)); note that $\Phi_1(v) = -\frac{1}{3} < 0 = v(1)$.

The terms after the summation sign in (3.2) are of the form $v(S \cup \{i\}) - v(S)$, where S is a subset of N not containing i. Note that there are exactly $|S|!\,(n - 1 - |S|)!$ orderings for which one has $P^{\,\sigma}(i) = S$. The first factor $|S|!$ corresponds to the number of orderings of S and the second factor $(n - 1 - |S|)!$ corresponds to the number of orderings of $N \setminus (S \cup \{i\})$. Using this, we can rewrite (3.2) and obtain

$$\Phi_i(v) = \sum_{S : i \notin S} \frac{|S|!\,(n - 1 - |S|)!}{n!} \left(v(S \cup \{i\}) - v(S) \right). \tag{3.3}$$

Note that $\frac{|S|!(n-1-|S|)!}{n!} = \frac{1}{n} \binom{n-1}{|S|}^{-1}$. This gives rise to a second probabilistic interpretation of the Shapley value. Create a subset S with $i \notin S$ in the following way. First, draw at random a number out of the urn consisting of possible sizes $0, \ldots, n-1$, where each number (i.e. size) has probability $\frac{1}{n}$ to be drawn. If size s is chosen, draw a set out of the urn consisting of subsets of $N \setminus \{i\}$ of size s, where each set has the same probability $\binom{n-1}{s}^{-1}$ to be drawn. If S is drawn, then one pays player i the amount $v(S \cup \{i\}) - v(S)$. Then, obviously in view of (3.3), the expected payoff for player i in this random procedure is the Shapley value for player i in the game $v \in G^N$.

Example 3.4. (i) For $v \in G^{\{1,2\}}$ we have

$$\Phi_i(v) = v(i) + \frac{v(1,2) - v(1) - v(2)}{2} \text{ for each } i \in \{1, 2\}.$$

(ii) The Shapley value $\Phi(v)$ for an additive game $v \in G^N$ is equal to

$(v(1), \ldots, v(n))$.
(iii) Let u_S be the unanimity game for $S \subset N$ (cf. (1.1)). Then $\Phi(u_S) = \frac{1}{|S|} e^S$.

The Shapley value satisfies some reasonable properties as introduced in Definition 1.25. More precisely

Proposition 3.5. *The Shapley value satisfies additivity, anonymity, the dummy player property, and efficiency.*

Proof. (*Additivity*) This follows from the fact that $m^\sigma(v+w) = m^\sigma(v) + m^\sigma(w)$ for all $v, w \in G^N$.

(*Anonymity*) We divide the proof into two parts.

(a) First we show that

$$\rho^* (m^\sigma(v)) = m^{\rho\sigma} (v^\rho) \text{ for all } v \in G^N \text{ and all } \rho, \sigma \in \pi(N).$$

This follows because for all $i \in N$:

$$\begin{aligned} (m^{\rho\sigma} (v^\rho))_{\rho\sigma(i)} &= v^\rho (\rho\sigma(1), \ldots, \rho\sigma(i)) - v^\rho (\rho\sigma(1), \ldots, \rho\sigma(i-1)) \\ &= v (\sigma(1), \ldots, \sigma(i)) - v (\sigma(1), \ldots, \sigma(i-1)) \\ &= (m^\sigma (v))_{\sigma(i)} = \rho^* (m^\sigma(v))_{\rho\sigma(i)} . \end{aligned}$$

(b) Take $v \in G^N$ and $\rho \in \pi(N)$. Then, using (a), the fact that $\rho \to \rho\sigma$ is a surjection on $\pi(N)$ and the linearity of ρ^*, we obtain

$$\begin{aligned} \Phi (v^\rho) &= \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^\sigma(v^\rho) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^{\rho\sigma}(v^\rho) \\ &= \frac{1}{n!} \sum_{\sigma \in \pi(N)} \rho^* (m^\sigma(v)) = \rho^* \left(\frac{1}{n!} \sum_{\sigma \in \pi(N)} m^\sigma(v) \right) \\ &= \rho^* (\Phi(v)) . \end{aligned}$$

This proves the anonymity of Φ.

(*Dummy player property*) This follows from (3.3) by noting that

$$\sum_{S: i \notin S} \frac{|S|! \, (n-1-|S|)!}{n!} = 1.$$

(*Efficiency*) Note that Φ is a convex combination of m^σ's and $\sum_{i \in N} m_i^\sigma(v) = v(N)$ for each $\sigma \in \pi(N)$.

By using the properties listed in Proposition 3.5 one can provide an axiomatic characterization of the Shapley value.

Theorem 3.6. *([58]) A solution $f : G^N \to \mathbb{R}^n$ satisfies additivity, anonymity, the dummy player property, and efficiency if and only if it is the Shapley value.*

Proof. In view of Proposition 3.5 we have only to show that if f satisfies the four properties, then $f = \Phi$.

Take $v \in G^N$. Then $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T$ with u_T being the unanimity game for coalition $T \in 2^N \setminus \{\emptyset\}$ and

$$c_T = \sum_{S : S \subset T} (-1)^{|T|-|S|} v(S)$$

(cf. (1.2)). Then by additivity we have $f(v) = \sum_{T \in 2^N \setminus \{\emptyset\}} f(c_T u_T)$, $\Phi(v) = \sum_{T \in 2^N \setminus \{\emptyset\}} \Phi(c_T u_T)$. So we have only to show that for all $T \in 2^N \setminus \{\emptyset\}$ and $c \in \mathbb{R}$:

$$f(cu_T) = \Phi(cu_T) . \tag{3.4}$$

Take $T \in 2^N \setminus \{\emptyset\}$ and $c \in \mathbb{R}$. Note first that for all $i \in N \setminus T$:

$$cu_T(S \cup \{i\}) - cu_T(S) = 0 = cu_T(i) \text{ for all } S \in 2^N \setminus \{\emptyset\} .$$

So, by the dummy player property, we have

$$f_i(cu_T) = \Phi_i(cu_T) = 0 \text{ for all } i \in N \setminus T. \tag{3.5}$$

Now suppose that $i, j \in T$, $i \neq j$. Then there is a $\sigma \in \pi(N)$ with $\sigma(i) = j$, $\sigma(j) = i$, $\sigma(k) = k$ for $k = N \setminus \{i, j\}$. It easily follows that $cu_T = \sigma(cu_T)$. Then anonymity implies that $\Phi(cu_T) = \Phi(\sigma(cu_T)) = \sigma^* \Phi(cu_T)$, $\Phi_{\sigma(i)}(cu_T) = \Phi_i(cu_T)$. So

$$\Phi_i(cu_T) = \Phi_j(cu_T) \text{ for all } i, j \in T, \tag{3.6}$$

and similarly $f_i(cu_T) = f_j(cu_T)$ for all $i, j \in T$.

Then efficiency, (3.5) and (3.6) imply that

$$f_i(cu_T) = \Phi_i(cu_T) = \frac{c}{|T|} \text{ for all } i \in T. \tag{3.7}$$

Now (3.5) and (3.7) imply (3.4). So $f(v) = \Phi(v)$ for all $v \in G^N$.

For other axiomatic characterizations of the Shapley value the reader is referred to [32], [43], and [77].

An alternative formula for the Shapley value is in terms of dividends (cf. [31]). The *dividends* d_T for each nonempty coalition T in a game $v \in G^N$ are defined in a recursive manner as follows:

$$d_T(v) := v(T) \text{ for all } T \text{ with } |T| = 1,$$

$$d_T(v) := \frac{v(T) - \sum_{S \subset T, S \neq T} |S| \, d_S(v)}{|T|} \text{ if } |T| > 1.$$

The relation between dividends and the Shapley value is described in the next theorem. It turns out that the Shapley value of a player in a game is the sum of all dividends of coalitions to which the player belongs.

Theorem 3.7. *Let $v \in G^N$ and $v = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T$. Then*
(i) $|T|\, d_T(v) = c_T$ for all $T \in 2^N \setminus \{\emptyset\}$.
(ii) $\Phi_i(v) = \sum_{T: i \in T} d_T(v)$ for all $i \in N$.

Proof. We have seen in the proof of Theorem 3.6 that $\Phi\left(c_T u_T\right) = \frac{c_T}{|T|} e^T$ for each $T \in 2^N \setminus \{\emptyset\}$, so by additivity,

$$\Phi\left(v\right) = \sum_{T \in 2^N \setminus \{\emptyset\}} \frac{c_T}{|T|} e^T.$$

Hence, $\Phi_i\left(v\right) = \sum_{T: i \in T} \frac{c_T}{|T|}$. The only thing we have to show is that

$$\frac{c_T}{|T|} = d_T \text{ for all } T \in 2^N \setminus \{\emptyset\}. \tag{3.8}$$

We prove this by induction. If $|T| = 1$, then $c_T = v(T) = d_T(v)$. Suppose (3.8) holds for all $S \subset T$, $S \neq T$. Then $|T|\, d_T(v) = v(T) - \sum_{S \subset T, S \neq T} |S|\, d_S(v) = v(T) - \sum_{S \subset T, S \neq T} c_S = c_T$ because $v(T) = \sum_{S \subset T} c_S$.

Now we turn to the description of the Shapley value by means of the *multilinear extension* of a game (cf. [49] and [50]).

Let $v \in G^N$. Consider the function $f : [0,1]^n \to \mathbb{R}$ on the hypercube $[0,1]^n$ defined by

$$f\left(x_1, \ldots, x_n\right) = \sum_{S \in 2^N} \left(\prod_{i \in S} x_i \prod_{i \in N \setminus S} \left(1 - x_i\right) \right) v(S). \tag{3.9}$$

In view of Theorem 1.31, the set of extreme points of $[0,1]^n$ is equal to $\left\{e^S \mid S \in 2^N\right\}$.

Proposition 3.8. *Let $v \in G^N$ and f be as above. Then $f\left(e^S\right) = v(S)$ for each $S \in 2^N$.*

Proof. Note that $\prod_{i \in S} \left(e^T\right)^i \prod_{i \in N \setminus S} \left(1 - \left(e^T\right)^i\right) = 1$ if $S = T$ and the product is equal to 0 otherwise. Then by (3.9) we have

$$f\left(e^T\right) = \sum_{S \in 2^N} \left(\prod_{i \in S} \left(e^T\right)^i \prod_{i \in N \setminus S} \left(1 - \left(e^T\right)^i\right) \right) v(S) = v(T).$$

One can give a probabilistic interpretation of $f(x)$. Suppose that each player $i \in N$, independently, decides whether to cooperate (with probability x_i) or not (with probability $1 - x_i$). Then with probability

$$\prod_{i \in S} x_i \prod_{i \in N \setminus S} \left(1 - x_i\right)$$

the coalition S forms, which has worth $v(S)$. Consequentially, $f(x)$ as given in (3.9) can be seen as the expectation of the worth of the formed coalition.

We denote by $D_k f(x)$ the derivative of f with respect to the k-th coordinate of x. Then we have the following result, describing the Shapley value $\Phi_k(v)$ of a game $v \in G^N$ as the integral along the main diagonal of $[0,1]^n$ of $D_k f$.

Theorem 3.9. *([49]) Let $v \in G^N$ and f be defined as in (3.9). Then $\Phi_k(v) = \int_0^1 (D_k f)(t,\ldots,t)\,dt$ for each $k \in N$.*

Proof. Note that

$$
\begin{aligned}
D_k f(x) &= \sum_{T: k\in T} \left(\prod_{i\in T\setminus\{k\}} x_i \prod_{i\in N\setminus T} (1-x_i) \right) v(T) \\
&\quad - \sum_{S: k\notin S} \left(\prod_{i\in S} x_i \prod_{i\in N\setminus(S\cup\{k\})} (1-x_i) \right) v(S) \\
&= \sum_{S: k\notin S} \left(\prod_{i\in S} x_i \prod_{i\in N\setminus(S\cup\{k\})} (1-x_i) \right) \left(v\left(S\cup\{k\}\right) - v(S) \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_0^1 (D_k f)(t,t,\ldots,t)\,dt = \\
&\sum_{S: k\notin S} \left(\int_0^1 t^{|S|} (1-t)^{n-|S|-1}\,dt \right) \left(v\left(S\cup\{k\}\right) - v(S) \right).
\end{aligned}
$$

Using the well known (beta)-integral formula

$$
\int_0^1 t^{|S|} (1-t)^{n-|S|-1}\,dt = \frac{|S|!\,(n-1-|S|)!}{n!}
$$

we obtain by (3.3)

$$
\begin{aligned}
\int_0^1 (D_k f)(t,t,\ldots,t)\,dt &= \sum_{S: k\notin S} \frac{|S|!\,(n-1-|S|)!}{n!} \left(v\left(S\cup\{k\}\right) - v(S) \right) \\
&= \Phi_k(v).
\end{aligned}
$$

Example 3.10. Let $v \in G^{\{1,2,3\}}$ with $v(1) = v(2) = v(1,2) = 0$, $v(1,3) = 1$, $v(2,3) = 2$, $v(N) = 4$. Then $f(x_1,x_2,x_3) = x_1(1-x_2)x_3 + 2(1-x_1)x_2x_3 + 4x_1x_2x_3 = x_1x_3 + 2x_2x_3 + x_1x_2x_3$ for all $x_1,x_2,x_3 \in [0,1]$. So $D_1 f(x) = x_3 + x_2x_3$. By Theorem 3.9 we obtain

$$
\Phi_1(v) = \int_0^1 D_1 f(t,t,t)\,dt = \int_0^1 \left(t + t^2\right) dt = \frac{5}{6}.
$$

3.2 The τ-value

The τ-value was introduced in [64] and it is defined for each *quasi-balanced game*. This value is based on the upper vector $M(N,v)$ and the lower vector $m(v)$ of a game $v \in G^N$ (cf. Section 2.2).

Definition 3.11. *A game $v \in G^N$ is called* ***quasi-balanced*** *if*
(i) $m(v) \leq M(N,v)$ and
(ii) $\sum_{i=1}^n m_i(v) \leq v(N) \leq \sum_{i=1}^n M_i(N,v)$.

The set of $|N|$-person quasi-balanced games will be denoted by Q^N.

Proposition 3.12. *If $v \in G^N$ is balanced, then $v \in Q^N$.*

Proof. Let $v \in G^N$ be balanced. Then, by Theorem 2.4, it has a non-empty core.

Let $x \in C(v)$. By Theorem 2.15 we have $m(v) \leq x \leq M(N,v)$. From this it follows $m(v) \leq M(N,v)$ and

$$\sum_{i=1}^n m_i(v) \leq \left(\sum_{i=1}^n x_i =\right) v(N) \leq \sum_{i=1}^n M_i(N,v).$$

Hence, $v \in Q^N$.

Definition 3.13. *For a game $v \in Q^N$ the τ-**value** $\tau(v)$ is defined by*

$$\tau(v) := \alpha m(v) + (1-\alpha)\, M(N,v)$$

where $\alpha \in [0,1]$ is uniquely determined by $\sum_{i \in N} \tau_i(v) = v(N)$.

Example 3.14. Let $v \in G^{\{1,2,3\}}$ with $v\,(N) = 5$, $v(i) = 0$ for all $i \in N$, $v\,(1,2) = v\,(1,3) = 2$, and $v\,(2,3) = 3$. Then $M(N,v) = (2,3,3)$, $m_1(v) = \max\{0,-1,-1,-1\} = 0$, $m_2(v) = m_3(v) = \max\{0,0,0,0\} = 0$. So $m(v) = 0$ and $v \in Q^{\{1,2,3\}}$. Hence, $\tau(v) = \alpha m(v) + (1-\alpha)\, M(N,v) = \frac{5}{8}\,(2,3,3) = \frac{5}{8} M(N,v)$.

Proposition 3.15. *Let $v \in Q^{\{1,2\}}$. Then*
(i) $C(v) = I(v)$,
(ii) $\tau(v) = \Phi(v)$,
(ii) $\tau(v)$ is in the middle of the core $C(v)$.

Proof. (i) For the lower and upper vectors we have

$$\begin{aligned} m_1(v) &= \max\{v\,(1)\,, v\,(1,2) - M_2(N,v)\} \\ &= \max\{v\,(1)\,, v\,(1,2) - (v\,(1,2) - v\,(1))\} \\ &= v\,(1)\,, \\ M_1(N,v) &= v\,(1,2) - v\,(2)\,. \end{aligned}$$

From $v \in Q^{\{1,2\}}$ it follows $v(1) = m_1(v) \leq M_1(N,v) = v(1,2) - v(2)$, i.e. v is superadditive and its imputation set $I(v)$ is non-empty. Then

$$C(v) = \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subset N \right\} = I(v).$$

(ii) For the Shapley value and for the τ-value we have $\Phi(v) = (\Phi_i(v))_{i \in \{1,2\}}$ with $\Phi_i(v) = \frac{1}{2} v(i) + \frac{1}{2}(v(1,2) - v(3-i))$, and

$$\begin{aligned} \tau(v) &= \frac{1}{2}(M(N,v) + m(v)) \\ &= \frac{1}{2}((v(1,2) - v(2), v(1,2) - v(1)) + v(1), v(2)) \\ &= \Phi(v). \end{aligned}$$

(iii) From (ii) it follows that $\Phi(v) = \tau(v) = \frac{1}{2}(f^1 + f^2)$ (cf. (2.1)), which is in the middle of the core $C(v)$.

Example 3.16. Let v be the 99-person game with
$v(N) = 1$, $v(S) = \frac{1}{2}$ if $\{1,2\} \subset S \neq N$,
$v(2,3,4,\ldots,99) = v(1,3,4,\ldots,99) = \frac{1}{4}$, and $v(S) = 0$ otherwise.
For the upper and lower vectors we have

$$M(N,v) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \ldots, \frac{1}{2} \right)$$

and

$$m(v) = (0, \ldots, 0).$$

So, $\tau(v) = (1-\alpha) M(N,v)$ with $1 - \alpha = \frac{4}{200}$. Hence,

$$\tau(v) = \frac{4}{200} \left(\frac{3}{4}, \frac{3}{4}, \frac{2}{4}, \ldots, \frac{2}{4} \right) = \frac{1}{200}(3,3,2,\ldots,2).$$

Remark 3.17. The game in Example 3.16 shows that the τ-value may not be in the core $C(v)$ of a game: note that $\tau_1(v) + \tau_2(v) = \frac{6}{200} < \frac{1}{2} = v(1,2)$.

Remark 3.18. For an axiomatic characterization of the τ-value the reader is referred to [65].

4

Classes of cooperative crisp games

In this chapter we consider three classes of cooperative crisp games: totally balanced games, convex games, and clan games. We introduce basic characterizations of these games and discuss special properties of the set-valued and one-point solution concepts introduced so far. Moreover, we relate the corresponding games with the concept of a population monotonic allocation scheme as introduced in [63]. We present the notion of a bi-monotonic allocation scheme for totally clan games and the constrained egalitarian solution (cf. [25] and [26]) for convex games.

4.1 Totally balanced games

4.1.1 Basic characterizations

Let $v \in G^N$. The game v is called *totally balanced* if all its subgames (cf. Definition 1.9) are balanced (cf. Definition 1.17). Equivalently, the game v is totally balanced if $C(v_T) \neq \emptyset$ for all $T \in 2^N \setminus \{\emptyset\}$ (cf. Theorem 2.4).

Example 4.1. Let $v \in G^{\{1,2,3,4\}}$ with $v(S) = 0, 0, 1, 2$ if $|S| = 0, 1, 3, 4$ respectively, and $v(1,2) = v(1,3) = v(2,3) = 1$, $v(1,4) = v(2,4) = v(3,4) = 0$. Then $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in C(v)$, so v is balanced, but the subgame v_T with $T = \{1,2,3\}$ is not balanced, so the game v is not totally balanced.

Example 4.2. Let $v \in G^{\{1,2,3,4\}}$ with $v(1,2) = v(3,4) = \frac{1}{2}$, $v(1,2,3) = v(2,3,4) = v(1,2,4) = v(1,3,4) = \frac{1}{2}$, $v(1,2,3,4) = 1$, and $v(S) = 0$ for all other $S \in 2^N$. Then $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \in C(v)$. Furthermore, $\left(0, \frac{1}{2}, 0\right)$ is an element of the core of the 3-person subgames and it easily follows that also the one- and two-person games have non-empty cores. Hence, the game v is totally balanced.

The following theorem relates totally balanced games that are non-negative (cf. Definition 1.11) and additive games (cf. Definition 1.12).

Theorem 4.3. *([38]) Let $v \in G^N$ be totally balanced and non-negative. Then v is the intersection of $2^n - 1$ additive games.*

Proof. Let $v \in G^N$ be as above. For each $T \in 2^N \setminus \{\emptyset\}$ consider the corresponding subgame v_T and take $x_T \in C(v_T)$. Define $y_T \in \mathbb{R}^n$ by $y_T^i := x_T^i$ if $i \in T$ and $y_T^i := \alpha$ if $i \in N \setminus T$, where $\alpha := \max_{S \in 2^N} v(S)$. We prove that v is equal to $\wedge_{T \in 2^N \setminus \{\emptyset\}} w_T$ $\left(:= \min \left\{ w_T \mid T \in 2^N \setminus \{\emptyset\} \right\}\right)$, where w_T is the additive game with $w_T(i) = y_T^i$ for all $i \in N$.

We have to show that for $S \in 2^N \setminus \{\emptyset\}$,

$$\min \left\{ w_T(S) \mid T \in 2^N \setminus \{\emptyset\} \right\} = v(S).$$

This follows from

(a) $w_S(S) = \sum_{i \in S} y_S^i = \sum_{i \in S} x_S^i = v_S(S) = v(S)$,

(b) $w_T(S) \geq \alpha \geq v(S)$ if $S \setminus T \neq \emptyset$, where the first inequality follows from the non-negativity of the game,

(c) $w_T(S) = \sum_{i \in S} x_S^i \geq v_T(S) = v(S)$ if $S \subset T$.

Nice examples of totally balanced games are games arising from flow situations with dictatorial control. A flow situation consists of a directed network with two special nodes called the source and the sink. For each arc there are a capacity constraint and a constraint with respect to the allowance to use that arc. Furthermore, with the aid of a simple game (cf. Definition 1.3) for each arc, one can describe which coalitions are allowed to use the arc. These are the coalitions which are winning (cf. Definition 1.4). Such games are called *control games* in this context. The value of a coalition S is the maximal flow per unit of time through the network from source to sink, where one uses only arcs which are controlled by S. Clearly, a dictatorial control game is a control game in which the arcs are controlled by dictators (cf. Definition 1.6).

One can show (cf. [38]) that each flow game with dictatorial control is totally balanced and non-negative. The converse is also true as shown in

Theorem 4.4. *([38]) Let $v \in G^N$ be totally balanced and non-negative. Then v is a flow game with dictatorial control.*

Proof. The minimum $v \wedge w$ of two flow games $v, w \in G^N$ with dictatorial control is again such a flow game: make a series connection of the flow networks of v and w. Also an additive game v is a flow game with dictatorial control. Combining these facts with Theorem 4.3 completes the proof.

4.1.2 Totally balanced games and population monotonic allocation schemes

The class of totally balanced games includes the class of games with a *population monotonic allocation scheme* (pmas). The latter concept was introduced in [63]. The idea here is that because of the complexity of the coalition formation process, players may not necessarily achieve full efficiency (if the game is superadditive it is efficient for the players to form the grand coalition). In order to take the possibility of partial cooperation into account, a pmas specifies not only how to allocate $v(N)$ but also how to allocate the value $v(S)$ of every coalition $S \in 2^N \setminus \{\emptyset\}$. Moreover, it reflects the intuition that there is "strength in numbers": the share allocated to each member is nondecreasing in the coalition size.

Definition 4.5. *Let $v \in G^N$. A scheme $a = (a_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ of real numbers is a* ***population monotonic allocation scheme*** *(pmas) of v if*
(i) $\sum_{i \in S} a_{iS} = v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,
(ii) $a_{iS} \leq a_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$.

Definition 4.6. *Let $v \in G^N$. An imputation $b \in I(v)$ is* ***pmas extendable*** *if there exist a pmas $a = (a_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ such that $a_{iN} = b_i$ for each player $i \in N$.*

As it can be easily derived from Definition 4.5, a necessary condition for a game to posses a pmas is that the game is totally balanced. This condition is also a sufficient one for games with at most three players: one can easily show that every core element of such a game is pmas extendable. However, if the number of players is at least four, the existence of a pmas is not guaranteed as the next example shows (cf. [63]).

Example 4.7. Let $v \in G^{\{1,2,3,4\}}$ with $v(i) = 0$ for $i = 1, \ldots, 4$, $v(1,2) = v(3,4) = 0$, $v(1,3) = v(1,4) = v(2,3) = v(2,4) = 1$, $v(S) = 1$ for all S with $|S| = 3$, and $v(N) = 2$. The core of this game is the line segment joining $(0,0,1,1)$ and $(1,1,0,0)$. One can easily see that each subgame of this game has a nonempty core, i.e. the game is totally balanced. However, the game lacks a pmas as it can be shown by the following argument: every pmas must satisfy $a_{1N} \geq a_{1\{1,3,4\}} = 1$, $a_{2N} \geq a_{2\{2,3,4\}} = 1$, $a_{3N} \geq a_{3\{1,2,3\}} = 1$, and $a_{4N} \geq a_{4\{1,2,4\}} = 1$. Hence, $\sum_{i \in N} a_{iN} \geq 4$, which is not feasible.

For general necessary and sufficient conditions for a game to possess a pmas the reader is referred to [63].

4.2 Convex games

This class of cooperative games was introduced in [60]. As we shall see, convex games have nice properties: the core of such a game is the unique stable set and its extreme points can be easily described. Moreover, the Shapley value coincides with the barycenter of the core.

4.2.1 Basic characterizations

Definition 4.8. *A game $v \in G^N$ is called **convex** iff*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \text{ for all } S, T \in 2^N. \qquad (4.1)$$

In what follows the set of convex games on player set N will be denoted by CG^N.

In the next theorem we give five characterizations of convex games. Characterizations (ii) and (iii) show that for convex games the gain made when individuals or groups join larger coalitions is higher than when they join smaller coalitions. Characterizations (iv) and (v) deal with the relation between the core and the Weber set.

Theorem 4.9. *Let $v \in G^N$. The following five assertions are equivalent.*
(i) $v \in CG^N$;
(ii) For all $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$ we have

$$v(S_1 \cup U) - v(S_1) \leq v(S_2 \cup U) - v(S_2); \qquad (4.2)$$

(iii) For all $S_1, S_2 \in 2^N$ and $i \in N$ such that $S_1 \subset S_2 \subset N \setminus \{i\}$ we have

$$v(S_1 \cup \{i\}) - v(S_1) \leq v(S_2 \cup \{i\}) - v(S_2); \qquad (4.3)$$

(iv) All $n!$ marginal vectors $m^\sigma(v)$ of v are elements of the core $C(v)$ of v;
(v) $W(v) = C(v)$.

Proof. We show $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$, $(iv) \Rightarrow (v)$, $(v) \Rightarrow (i)$.

(a) Suppose that (i) holds. Take $S_1, S_2, U \in 2^N$ with $S_1 \subset S_2 \subset N \setminus U$. From (4.1) with $S_1 \cup U$ in the role of S and S_2 in the role of T we obtain (4.2) by noting that $S \cup T = S_2 \cup U$, $S \cap T = S_1$. Hence, (i) implies (ii).

(b) That (ii) implies (iii) is trivial (take $U = \{i\}$).

(c) Suppose that (iii) holds. Let $\sigma \in \pi(N)$ and take m^σ. Then $\sum_{k=1}^n m_k^\sigma = v(N)$. To prove that $m^\sigma \in C(v)$ we have to show that for $S \in 2^N$: $\sum_{k \in S} m_k^\sigma \geq v(S)$.

Let $S = \{\sigma(i_1), \ldots, \sigma(i_k)\}$ with $i_1 < \ldots < i_k$. Then

$$\begin{aligned} v(S) &= \sum_{r=1}^{k} \left(v(\sigma(i_1), \ldots, \sigma(i_r)) - v(\sigma(i_1), \ldots, \sigma(i_{r-1})) \right) \\ &\leq \sum_{r=1}^{k} \left(v(\sigma(1), \ldots, \sigma(i_r)) - v(\sigma(1), \ldots, \sigma(i_r - 1)) \right) \\ &= \sum_{r=1}^{k} m_{\sigma(i_r)}^\sigma = \sum_{k \in S} m_k^\sigma, \end{aligned}$$

where the inequality follows from (iii) applied to $i := \sigma(i_r)$ and $S_1 := \{\sigma(i_1), \ldots, \sigma(i_{r-1})\} \subset S_2 := \{\sigma(1), \ldots, \sigma(i_r - 1)\}$ for $r \in \{1, \ldots, k\}$. This proves that (iii) implies (iv).

(d) Suppose that (iv) holds. Since $C(v)$ is a convex set, we have $C(v) \supset co\{m^\sigma \mid \sigma \in \pi(N)\} = W(v)$. From Theorem 2.18 we know that $C(v) \subset W(v)$. Hence, (v) follows from (iv).

(e) Finally we prove that (v) implies (i). Take $S, T \in 2^N$. Then, there is $\sigma \in \pi(N)$ and $d, t, u \in \mathbb{N}$ with $0 \leq d \leq t \leq u \leq n$ such that $S \cap T = \{\sigma(i_1), \ldots, \sigma(d)\}$, $T \setminus S = \{\sigma(d+1), \ldots, \sigma(t)\}$, $S \setminus T = \{\sigma(t+1), \ldots, \sigma(u)\}$, $N \setminus (S \cup T) = \{\sigma(u+1), \ldots, \sigma(n)\}$. From (v) follows that $m^\sigma \in C(v)$; hence,

$$v(S) \leq \sum_{i \in S} m_i^\sigma. \tag{4.4}$$

On the other hand

$$\begin{aligned}\sum_{i \in S} m_i^\sigma &= \sum_{r=1}^{d} \left(v(A_r) - v(A_{r-1})\right) + \sum_{k=1}^{u-t} v(T \cup B_{t+k}) - v(T \cup B_{t+k-1}) \\ &= v(S \cap T) + v(S \cup T) - v(T),\end{aligned}$$

where $A_r = \{\sigma(1), \ldots, \sigma(r)\}$, $A_{r-1} = A_r \setminus \{\sigma(r)\}$, $B_{t+k} = \{\sigma(t+1), \ldots, \sigma(t+k)\}$, and $B_{t+k-1} = B_{t+k} \setminus \{\sigma(t+k)\}$.

Combining (4.4) and (4.5) yields (4.1). This completes the proof.

Remark 4.10. It follows easily from Theorem 4.9 that for each game $v \in CG^N$ we have that the Shapley value $\Phi(v)$ coincides with the barycenter of the core $C(v)$.

Definition 4.11. *([55]) A game $v \in G^N$ is called* ***exact*** *if for each $S \in 2^N \setminus \{\emptyset\}$ there is an $x \in C(v)$ with $\sum_{i \in S} x_i = v(S)$.*

Remark 4.12. It is not difficult to see that a convex game is exact.

With respect to stable sets for convex games we have

Theorem 4.13. *([60]) Let $v \in CG^N$. Then $C(v)$ is the unique stable set.*

Proof. In view of Theorem 2.11 we only have to show that $C(v)$ is stable. This is true if v is additive. So we suppose that v is not additive.

Let $y \in I(v) \setminus C(v)$. Take an $S \in 2^N \setminus \{\emptyset\}$ such that

$$\frac{v(S) - \sum_{i \in S} y_i}{|S|} = \max_{C \in 2^N \setminus \{\emptyset\}} \frac{v(C) - \sum_{i \in C} y_i}{|C|}. \tag{4.5}$$

Further take $z \in C(v)$ such that $\sum_{i \in S} z_i = v(S)$. This is possible in view of Remark 4.12.

Let $x \in \mathbb{R}^n$ be the vector with

$$x_i := \begin{cases} y_i + \frac{v(S)-\sum_{i\in S} y_i}{|S|} & \text{if } i \in S, \\ z_i & \text{otherwise.} \end{cases}$$

Then $x \in I(v)$ and $x \operatorname{dom}_S y$. To prove that $x \in C(v)$, note first of all that for $T \in 2^N$ with $T \cap S \neq \emptyset$ we have

$$\begin{aligned}
\sum_{i\in T\cap S} x_i &= \sum_{i\in T\cap S} (x_i - y_i) + \sum_{i\in T\cap S} y_i \\
&= |T \cap S| \, \frac{v(S) - \sum_{i\in S} y_i}{|S|} + \sum_{i\in T\cap S} y_i \\
&\geq \left(v(T\cap S) - \sum_{i\in T\cap S} y_i \right) + \sum_{i\in T\cap S} y_i \\
&= v(T \cap S),
\end{aligned}$$

where the inequality follows from (4.5).

But then

$$\begin{aligned}
\sum_{i\in T} x_i &= \sum_{i\in T\cap S} x_i + \sum_{i\in T\setminus S} z_i \\
&\geq v(T\cap S) + \sum_{i\in T\cup S} z_i - \sum_{i\in S} z_i \\
&\geq v(T\cap S) + v(T\cup S) - v(S) \\
&\geq v(T)
\end{aligned}$$

because $z \in C(v)$, $\sum_{i\in S} z_i = v(S)$ and $v \in CG^N$.

For $T \in 2^N \setminus \{\emptyset\}$ with $T \cap S = \emptyset$ we have $\sum_{i\in T} x_i = \sum_{i\in T} z_i \geq v(T)$ because $z \in C(v)$. So we have proved that $x \in C(v)$.

Then $I(v) = C(v) \cup dom\,(C(v))$ and $C(v) \cap dom(C(v)) = \emptyset$. Hence, $C(v)$ is a stable set.

4.2.2 Convex games and population monotonic allocation schemes

As we have pointed out in Section 4.1.2, a necessary condition for the existence of a pmas (cf. Definition 4.5) is the total balancedness of the game. A sufficient condition for the existence of a pmas is the convexity of the game. In order to see this we will need the following definitions.

For all $\rho \in \pi(N)$ and all $i \in N$, let

$$N(\rho, i) = \{ j \in N \mid \rho\,(j) \leq \rho(i) \}\,.$$

One generalizes the definition of a marginal contribution vector (cf. Definition 1.8) as follows.

Definition 4.14. *Let $v \in G^N$ and $\rho \in \pi(N)$. The* ***extended vector of marginal contributions*** *associated with ρ is the vector $a^\rho = (a^\rho_{iS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ defined component-wise by*

$$a^\rho_{iS} = v\left(N(\rho, i) \cap S\right) - v\left((N(\rho, i) \cap S) \setminus \{i\}\right).$$

Proposition 4.15. *([63]) Let $v \in CG^N$. Then every extended vector of marginal contributions is a pmas for v.*

Proof. Take $v \in CG^N$, $\rho \in \pi(N)$, and a^ρ. Pick an arbitrary $S \in 2^N \setminus \{\emptyset\}$ and rank all players $i \in S$ in increasing order of $\rho(i)$. Let $i, i' \in S$ be two players such that i' immediately follows i. Observe that

$$a^\rho_{iS} = v\left(N(\rho, i) \cap S\right) - v\left((N(\rho, i) \cap S) \setminus \{i\}\right),$$

and

$$\begin{aligned} a^\rho_{i'S} &= v\left(N(\rho, i') \cap S\right) - v\left((N(\rho, i') \cap S) \setminus \{i'\}\right) \\ &= v\left(N(\rho, i') \cap S\right) - v\left((N(\rho, i) \cap S)\right). \end{aligned}$$

Therefore, $a^\rho_{iS} + a^\rho_{i'S} = v\left(N(\rho, i') \cap S\right) - v\left((N(\rho, i) \cap S) \setminus \{i\}\right)$. Repeating this argument leads to $\sum_{i \in S} a^\rho_{iS} = v(S)$, which establishes the feasibility of a^ρ.

As for the monotonicity property in Definition 4.5, note that if $i \in S \subset T \subset N$, then $S \cap N(\rho, i) \subset T \cap N(\rho, i)$ for all $i \in N$. Hence, by the convexity of v we have $a^\rho_{iS} \leq a^\rho_{iT}$. This completes the proof.

According to [60] the core of a convex game is a polytope whose extreme points are the (usual) marginal contribution vectors (cf. Theorem 4.9). Because every convex combination of pmas of a game v is itself a pmas for that game, one obtains

Proposition 4.16. *Let $v \in CG^N$ and $b = (b_i)_{i \in N} \in C(v)$. Then b is pmas extendable.*

Definition 4.17. *Let $v \in G^N$. The* ***extended Shapley value*** *of v is the vector $\widetilde{\Phi}(v)$ defined component-wise as follows: for all $S \in 2^N \setminus \{\emptyset\}$ and all $i \in S$,*

$$\widetilde{\Phi}_{iS}(v) = \Phi_i(v_S),$$

where $\Phi(v_S) = (\Phi_i(v_S))_{i \in S}$ is the Shapley value of the game v_S.

As shown in [63], the extended Shapley value is the aritmethic average of the extended vectors of marginal contributions (cf. Definition 4.14). Therefore, one obtains

Proposition 4.18. *Let $v \in CG^N$. Then the extended Shapley value of v is a pmas for v.*

4.2.3 The constrained egalitarian solution for convex games

Another interesting element of the core of a game $v \in CG^N$ is the constrained egalitarian allocation $E(v)$ introduced in [26] which can be described in a simple way and found easily in a finite number of steps. Two lemmas in which the average worth $\frac{v(S)}{|S|}$ of a nonempty coalition S with respect to the characteristic function v plays a role, are used further.

Lemma 4.19. *Let $v \in CG^N$ and $L(v) := \arg\max_{C \in 2^N \setminus \{\emptyset\}} \frac{v(C)}{|C|}$. Then*
(i) The set $L(v) \cup \{\emptyset\}$ is a lattice, i.e. for all $S_1, S_2 \in L(v) \cup \{\emptyset\}$ we have $S_1 \cap S_2 \in L(v) \cup \{\emptyset\}$ and $S_1 \cup S_2 \in L(v) \cup \{\emptyset\}$;
(ii) In $L(v)$ there is a maximal element with respect to $\subset$ namely

$$\cup \{S \mid S \in L(v)\}.$$

Proof. (i) Let $\alpha := \max_{C \in 2^N \setminus \{\emptyset\}} \frac{v(C)}{|C|}$ and suppose $\frac{v(S_1)}{|S_1|} = \alpha = \frac{v(S_2)}{|S_2|}$ for some $S_1, S_2 \in 2^N \setminus \{\emptyset\}$. We have to prove that

$$\frac{v(S_1 \cup S_2)}{|S_1 \cup S_2|} = \alpha \text{ and } v(S_1 \cap S_2) = \alpha \, |S_1 \cap S_2| \, . \tag{4.6}$$

We have

$$\begin{aligned} v(S_1 \cup S_2) + v\,(S_1 \cap S_2) &= \frac{v(S_1 \cup S_2)}{|S_1 \cup S_2|} \, |S_1 \cup S_2| + \frac{v(S_1 \cap S_2)}{|S_1 \cap S_2|} \, |S_1 \cap S_2| \\ &\leq \alpha \, |S_1 \cup S_2| + \alpha \, |S_1 \cap S_2| = \alpha \, |S_1| + \alpha \, |S_1| \\ &= v(S_1) + v(S_2) \leq v(S_1 \cup S_2) + v(S_1 \cap S_2), \end{aligned}$$

where the first inequality follows from the definition of α and the second inequality follows from $v \in CG^N$. So everywhere we have equalities, which proves (4.6).

(ii) This assertion follows immediately from (i) and the finiteness of $L(v)$.

Lemma 4.20. *Let $v \in CG^N$ and $S \subset N, S \neq N$. Then $v^{-S} \in CG^{N \setminus S}$, where*

$$v^{-S}\,(T) := v\,(S \cup T) - v(S) \textit{ for all } T \in 2^{N \setminus S}.$$

Proof. Let $T_1 \subset T_2 \subset (N \setminus S) \setminus \{i\}$ where $i \in N \setminus S$. We have to prove that $v^{-S}\,(T_1 \cup \{i\}) - v(T_1) \leq v^{-S}\,(T_2 \cup \{i\}) - v(T_2)$. Notice that this is equivalent to prove that $v\,(S \cup T_1 \cup \{i\}) - v(S \cup T_1) \leq v\,(S \cup T_2 \cup \{i\}) - v(S \cup T_2)$ which follows by the convexity of v.

Given these two lemmas, one can find the egalitarian allocation $E\,(v)$ of a game $v \in CG^N$ according to the following algorithm (cf. [26]).

In Step 1 of the algorithm one considers the game $\langle N_1, v_1 \rangle$ with $N_1 := N$, $v_1 := v$, and the per capita value $\frac{v_1(T)}{|T|}$ for each non-empty subcoalition T of N_1. Then the largest element $T_1 \in 2^{N_1} \setminus \{\emptyset\}$ in $\arg\max_{T \in 2^{N_1} \setminus \{\emptyset\}} \frac{v_1(T)}{|T|}$

is taken (such an element exists according to Lemma 4.19) and $E_i(N,v) = \frac{v_1(T)}{|T|}$ for all $i \in T_1$ is defined. If $T_1 = N$, then we stop.

In case $T_1 \neq N$, then in Step 2 of the algorithm one considers the convex game $\langle N_2, v_2 \rangle$ where $N_2 := N_1 \setminus T_1$ and $v_2(S) = v_1(S \cup T_1) - v_1(T_1)$ for each $S \in 2^{N_2} \setminus \{\emptyset\}$ (cf. Lemma 4.20) takes the largest element T_2 in $\arg\max_{T \in 2^{N_2} \setminus \{\emptyset\}} \frac{v_2(T)}{|T|}$ and defines $E_i(v) = \frac{v_2(T)}{|T|}$ for all $i \in T_2$. If $T_1 \cup T_2 = N$ we stop; otherwise we continue by considering the game $\langle N_3, v_3 \rangle$ with $N_3 := N_2 \setminus T_2$ and $v_3(S) = v_2(S \cup T_2) - v_2(T_2)$ for each $S \in 2^{N_3} \setminus \{\emptyset\}$, etc. After a finite number of steps the algorithm stops, and the obtained allocation $E(v) \in \mathbb{R}^n$ is called the *constrained egalitarian solution* of the game $v \in CG^N$.

Theorem 4.21. *Let $v \in CG^N$ and let $E(v)$ be the constrained egalitarian solution. Then $E(v) \in C(v)$.*

Proof. Suppose that $S_1, \ldots, S_m$ is the ordered partition of N on which $E(v)$ is based. So

$$E_i(v) = \frac{1}{|S_1|} v(S_1) \text{ if } i \in S_1,$$

and for $k \geq 2$:

$$E_i(v) = \frac{1}{|S_k|} \left(v\left(\cup_{r=1}^{k} S_r\right) - v\left(\cup_{r=1}^{k-1} S_r\right) \right) \quad \text{if } i \in S_k,$$

and for all $T \subset \cup_{r=k}^{m} S_r$ $(k \geq 1)$:

$$\frac{v\left(\left(\cup_{r=1}^{k-1} S_r\right) \cup T\right) - v\left(\cup_{r=1}^{k-1} S_r\right)}{|T|} \leq \frac{v\left(\cup_{r=1}^{k} S_r\right) - v\left(\cup_{r=1}^{k-1} S_r\right)}{|S_r|}. \tag{4.7}$$

First we prove that $E(v)$ is efficient, i.e. $\sum_{i=1}^{n} E_i(v) = v(N)$. This follows by noting that

$$\begin{aligned}
\sum_{i=1}^{n} E_i(N,v) &= \sum_{i \in S_1} E_i(v) + \sum_{k=2}^{m} \sum_{i \in S_k} E_i(v) \\
&= v(S_1) + \sum_{k=2}^{m} \left(v\left(\cup_{r=1}^{k} S_r\right) - v\left(\cup_{r=1}^{k-1} S_r\right) \right) \\
&= v\left(\cup_{r=1}^{m} S_r\right) = v(N).
\end{aligned}$$

Now we prove the stability of $E(v)$. Take $S \subset N$. We have to prove that $\sum_{i \in S} E_i(v) \geq v(S)$. Note first that $S = \cup_{r=1}^{m} T_r$, where $T_r := S \cap S_r$ $(r = 1, \ldots, m)$. Then

$$
\begin{aligned}
\sum_{i\in S} E_i(v) &= \sum_{i\in T_1} E_i(v) + \sum_{k=2}^{m} \sum_{i\in T_k} E_i(v) \\
&= |T_1| \frac{v(S_1)}{|S_1|} + \sum_{k=2}^{m} |T_k| \frac{\left(v\left(\cup_{r=1}^{k} S_r\right) - v\left(\cup_{r=1}^{k-1} S_r\right)\right)}{|S_k|} \\
&\geq |T_1| \frac{v(T_1)}{|T_1|} + \sum_{k=2}^{m} |T_k| \frac{\left(v\left(\left(\cup_{r=1}^{k-1} S_r\right) \cup T_k\right) - v\left(\cup_{r=1}^{k-1} S_r\right)\right)}{|T_k|} \\
&\geq v(T_1) + \sum_{k=2}^{m} \left(v\left(\left(\cup_{r=1}^{k-1} T_r\right) \cup T_k\right) - v\left(\cup_{r=1}^{k-1} T_r\right)\right) \\
&= v\left(\cup_{r=1}^{m} T_r\right) = v(S),
\end{aligned}
$$

where the first inequality follows from (4.7), and the second inequality follows by the convexity of v by noting that $\cup_{r=1}^{k-1} S_r \supset \cup_{r=1}^{k-1} T_r$ for all $k \in \{2, \ldots, m\}$.

Since the constrained egalitarian solution is in the core of the corresponding convex game, it has been interesting to study the interrelation between $E(v)$ and every other core allocation in terms of a special kind of domination which can be introduced as follows.

Consider a society of n individuals with aggregate income fixed at I units. For any $x \in \mathbb{R}_+^n$ denote by $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ the vector obtained by rearranging its coordinates in a non-decreasing order, that is, $\widehat{x}_1 \leq \widehat{x}_2 \leq \ldots \leq \widehat{x}_n$. For any $x, y \in \mathbb{R}_+^n$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = I$, we say that x *Lorenz dominates* y, and denote it by $x \succ_L y$, iff $\sum_{i=1}^p \widehat{x}_i \geq \sum_{i=1}^p \widehat{y}_i$ for all $p \in \{1, \ldots, n-1\}$, with at least one strict inequality.

It turns out that for convex games the constrained egalitarian solution Lorenz dominates every other core allocation; for a proof the reader is referred to [26].

4.3 Clan games

Clan games were introduced in [52] to model social conflicts between "powerful" players (clan members) and "powerless" players (non-clan members). In a clan game the powerful players have veto power and the powerless players operate more profitably in unions than on their own. Economic applications of such clan games to bankruptcy problems, production economies, and information acquisition are provided in [14], [44], and [52].

4.3.1 Basic characterizations

Definition 4.22. *A game $v \in G^N$ is a* ***clan game*** *with clan $C \in 2^N \setminus \{\emptyset, N\}$ if it satisfies the following four conditions:*

(a) Nonnegativity: $v(S) \geq 0$ *for all* $S \subset N$*;*
(b) Nonnegative marginal contributions to the grand coalition: $M_i(N, v) \geq 0$ *for each player* $i \in N$*;*
(c) Clan property: every player $i \in C$ *is a veto player, i.e.* $v(S) = 0$ *for each coalition* S *that does not contain* C*;*
(d) Union property: $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(N, v)$ *if* $C \subset S$*.*

If the clan consists of a single member, the corresponding game is called a *big boss game* (cf. [44]).

The next proposition shows that the core of a clan game has an interesting shape.

Proposition 4.23. *([52]) Let* $v \in G^N$ *be a clan game. Then*

$$C(v) = \{x \in I(v) \mid x_i \leq M_i(N, v) \text{ for all } i \in N \setminus C\}.$$

Proof. Suppose $x \in C(v)$. Then $\sum_{i \in N \setminus \{i\}} x_i \geq v(N \setminus \{i\})$ for all $i \in N \setminus C$. Since $v(N) = \sum_{i \in N} x_i = \sum_{j \in N \setminus \{i\}} x_j + x_i$ one has

$$x_i = v(N) - \sum_{j \in N \setminus \{i\}} x_j \leq v(N) - v(N \setminus \{i\}) = M_i(N, v) \text{ for all } i \in N \setminus C.$$

Conversely, if $x \in I(v)$ and $x_i \leq M_i(N, v)$ for all $i \in N \setminus C$, then, for a coalition S which does not contain C, one finds that $\sum_{i \in S} x_i \geq 0 = v(S)$. If $C \subset S$, then, by using condition (d) in Definition 4.22, one has

$$v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(N, v) \geq \sum_{i \in N \setminus S} x_i.$$

Since $v(N) = \sum_{i \in N} x_i$, one finally obtains $\sum_{i \in S} x_i \geq v(S)$, i.e. $x \in C(v)$.

In fact, a clan game can be fully described by the shape of the core as indicated in

Proposition 4.24. *([52]) Let* $v \in G^N$ *and* $v \geq 0$*. The game* v *is a clan game iff*
(i) $v(N)e^j \in C(v)$ *for all* $j \in C$*;*
(ii) There is at least one element $x \in C(v)$ *such that* $x_i = M_i(N, v)$ *for all* $i \in N \setminus C$*.*

Proof. One needs to prove only sufficiency. Suppose $S \in 2^N \setminus \{\emptyset\}$ does not contain C. Take $j \in C \setminus S$. Because $x := v(N)e^j \in C(v)$, one has $\sum_{i \in S} x_i = 0 \geq v(S)$ and from $v \geq 0$ one finds the clan property in Definition 4.22.

If $C \subset S$ and $x \in C(v)$ with $x_i = M_i(N, v)$ for all $i \in N \setminus C$, then

$$v(S) \leq \sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus S} x_i = v(N) - \sum_{i \in N \setminus S} M_i(N, v),$$

proving the union property in Definition 4.22.

Furthermore, $M_i(N, v) = x_i \geq v(i) = 0$ for all $i \in N \setminus C$.

4.3.2 Total clan games and monotonic allocation schemes

The subgames in a total clan game inherit the structure of the original (clan) game. This leads to the following

Definition 4.25. *A game $v \in G^N$ is a* ***total clan game*** *with clan $C \in 2^N \setminus \{\emptyset, N\}$ if v_S is a clan game (with clan C) for every coalition $S \supset C$.*

Note that in Definition 4.25 attention is restricted to coalitions that contain the clan C, since the clan property of v implies that in the other subgames the characteristic function is simply the zero function.

The next theorem provides a characterization of total clan games. The reader is referred to [73] for its proof.

Theorem 4.26. *Let $v \in G^N$ and $C \in 2^N \setminus \{\emptyset, N\}$. The following claims are equivalent:*
(i) v is a total clan game with clan C;
(ii) v is monotonic, every player $i \in C$ is a veto player, and for all coalitions S and T with $S \supset C$ and $T \supset C$:

$$S \subset T \text{ implies } v(T) - v(S) \geq \sum_{i \in T \setminus S} M_i(T, v); \tag{4.8}$$

(iii) v is monotonic, every player $i \in C$ is a veto player, and for all coalitions S and T with $S \supset C$ and $T \supset C$:

$$S \subset T \text{ and } i \in S \setminus C \text{ imply } M_i(S, v) \geq M_i(T, v). \tag{4.9}$$

One can study also the question whether a total clan game possesses a pmas (cf. Definitions 4.5, 4.6). The attention in [73] is restricted to the allocation of $v(S)$ for coalitions $S \supset C$, since other coalitions have value zero by the clan property.

Theorem 4.27. *([73]) Let $v \in G^N$ be a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$ and let $b \in C(v)$. Then b is pmas extendable.*

Proof. According to Proposition 4.23 we have

$$C(v) = \{x \in I(v) \mid x_i \leq M_i(N, v) \text{ for all } i \in N \setminus C\}.$$

Hence there exists, for each player $i \in N$, a number $\alpha_i \in [0, 1]$ such that $\sum_{i \in C} \alpha_i = 1$ and

$$b_i = \begin{cases} \alpha_i M_i(N, v) & \text{if } i \in N \setminus C, \\ \alpha_i \left[v(N) - \sum_{j \in N \setminus C} \alpha_j M_j(N, v) \right] & \text{if } i \in C. \end{cases}$$

In other words, each non-clan member receives a fraction of his marginal contribution to the grand coalition, whereas the clan members divide the remainder.

Define for each $S \supset C$ and $i \in S$:

$$a_{iS} = \begin{cases} \alpha_i M_i(N,v) & \text{if } i \in S \setminus C, \\ \alpha_i \left[v(N) - \sum_{j \in S \setminus C} \alpha_j M_j(N,v) \right] & \text{if } i \in C. \end{cases}$$

Clearly, $a_{iN} = b_i$ for each player $i \in N$. We proceed to prove that the vector $(a_{iS})_{i \in S, S \supset C}$ is a pmas. Since $\sum_{i \in C} \alpha_i = 1$, it follows that $\sum_{i \in S} a_{iS} = v(S)$. Now let $S \supset C, T \supset C$ and $i \in S \subset T$.

- If $i \notin C$, then $a_{iS} = a_{iT} = \alpha_i M_i(N,v)$.
- If $i \in C$, then

$$\begin{aligned} a_{iT} - a_{iS} &= \alpha_i \Big[v(T) - \sum_{j \in T \setminus C} \alpha_j M_j(N,v) \Big] - \alpha_i \Big[v(S) - \sum_{j \in S \setminus C} \alpha_j M_j(N,v) \Big] \\ &= \alpha_i \Big[v(T) - v(S) - \sum_{j \in T \setminus S} \alpha_j M_j(N,v) \Big] \\ &\geq \alpha_i \Big[v(T) - v(S) - \sum_{j \in T \setminus S} M_j(N,v) \Big] \\ &\geq \alpha_i \Big[v(T) - v(S) - \sum_{j \in T \setminus S} M_j(T,v) \Big] \\ &\geq 0, \end{aligned}$$

where the first inequality follows from nonnegativity of the marginal contributions, the second inequality follows from (4.9), and the final inequality from (4.8). Consequently, $(a_{iS})_{i \in S, S \supset C}$ is a pmas.

Whereas a pmas allocates a larger payoff to each player as the coalitions grow larger, property (4.9) suggests a slightly different approach in total clan games: the marginal contribution of each non-clan member actually decreases in a larger coalition. Taking this into account, one might actually allocate a smaller amount to the non-clan members in larger coalitions. Moreover, to still maintain some stability, such allocations should still give rise to core allocations in the subgames. An allocation scheme that satisfies these properties is called *bi-monotonic allocation scheme* (cf. [73]).

Definition 4.28. *Let $v \in G^N$ be a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$. A* ***bi-monotonic allocation scheme*** *(bi-mas) for the game v is a vector $a = (a_{iS})_{i \in S, S \supset C}$ of real numbers such that*
(i) $\sum_{i \in S} a_{iS} = v(S)$ for all $S \in 2^C \setminus \{\emptyset\}$,
(ii) $a_{iS} \leq a_{iT}$ for all $S \supset C, T \supset C$ with $S \subset T$ and $i \in S \cap C$,
(iii) $a_{iS} \geq a_{iT}$ for all $S \supset C, T \supset C$ with $S \subset T$ and $i \in S \setminus C$,
(iv) $(a_{iS})_{i \in S}$ is a core element of the subgame v_S for each coalition $S \supset C$.

Definition 4.29. *Let $v \in G^N$ be a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$. An imputation $b \in I(v)$ is* ***bi-mas extendable*** *if there exist a bi-mas $a = (a_{iS})_{i \in S, S \supset C}$ such that $a_{iN} = b_i$ for each player $i \in N$.*

Theorem 4.30. *([73]) Let $v \in G^N$ be a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$ and let $b \in C(v)$. Then b is bi-mas extendable.*

Proof. Take $(\alpha_i)_{i\in N} \in [0,1]^N$ as in the proof of Theorem 4.27. Define for each $S \supset C$ and $i \in S$:

$$a_{iS} = \begin{cases} \alpha_i M_i(S,v) & \text{if } i \in S \setminus C, \\ \alpha_i \left[v(S) - \sum_{j\in S\setminus C} \alpha_j M_j(S,v) \right] & \text{if } i \in C. \end{cases}$$

We proceed to prove that $(a_{iS})_{i\in S, S\supset C}$ is a bi-mas. Since $\sum_{i\in C} \alpha_i = 1$, it follows that $\sum_{i\in S} a_{iS} = v(S)$. Now let $S \supset C, T \supset C$ and $i \in S \subset T$.

- If $i \in N \setminus C$, then $a_{iS} = \alpha_i M_i(S,v) \geq \alpha_i M_i(T,v) = a_{iT}$ by (4.9).
- If $i \in C$, then

$$\begin{aligned} a_{iT} - a_{iS} &= \alpha_i \Big[v(T) - \sum_{j\in T\setminus C} \alpha_j M_j(T,v) \Big] \\ &\quad - \alpha_i \Big[v(S) - \sum_{j\in S\setminus C} \alpha_j M_j(S,v) \Big] \\ &= \alpha_i \Big[v(T) - v(S) - \sum_{j\in T\setminus S} \alpha_j M_j(T,v) \Big] \\ &\quad + \alpha_i \Big[\sum_{j\in S\setminus C} \alpha_j \left(M_j(S,v) - M_j(T,v) \right) \Big] \\ &\geq \alpha_i \Big[v(T) - v(S) - \sum_{j\in T\setminus S} \alpha_j M_j(T,v) \Big] \\ &\geq 0, \end{aligned}$$

  where the first inequality follows from (4.8) and nonnegativity of $(\alpha_j)_{j\in T\setminus S}$, and the second inequality follows from (4.9).

Finally, for each coalition $S \supset C$, the vector $(a_{iS})_{i\in S, S\supset C}$ is shown to be a core allocation of the clan game v_S. Let $S \supset C$. According to Proposition 4.23 we have

$$C(v) = \{ x \in I(v) \mid x_i \leq M_i(N,v) \text{ for all } i \in N \setminus C \}.$$

Let $i \in S\setminus C$. Then $a_{iS} = \alpha_i M_i(S,v) \leq M_i(S,v)$. Also, $\sum_{i\in S} a_{iS} = v(S)$, so $(a_{iS})_{i\in S}$ satisfies efficiency.

To prove individual rationality, consider the following three cases:

- Let $i \in S \setminus C$. Then $a_{iS} = \alpha_i M_i(S,v) \geq 0 = v(i)$;
- Let $i \in S \cap C$ and $|C| = 1$. Then $C = \{i\}$ and by construction $\alpha_i = \sum_{j\in C} \alpha_j = 1$. Hence $a_{iS} \geq a_{iC} = \alpha_i v(C) = v(i)$;
- Let $i \in S \cap C$ and $|C| > 1$. Then $a_{iS} \geq a_{iC} = \alpha_i v(C) \geq 0 = v(i)$, since every player in C is a veto player.

Consequently, $(a_{iS})_{i\in S, S\supset C}$ is a bi-mas.

Part II

Cooperative games with fuzzy coalitions

Cooperative games with fuzzy coalitions are introduced in [1] and [2]. Such games are helpful for approaching sharing problems arising from economic situations where agents have the possibility to cooperate with different participation levels, varying from non-cooperation to full cooperation, and where the obtained reward depends on the levels of participation. A fuzzy coalition describes the participation levels to which each player is involved in cooperation. For example, in a class of production games, partial participation in a coalition means to offer a part of the available resources while full participation means to offer all the resources. Since in classical cooperative games agents are either fully involved or not involved at all in cooperation with some other agents, one can look at the classical games as a simplified version of games with fuzzy coalitions. A fuzzy game is represented by a real valued function that assigns a real number to each fuzzy coalition.

Games with fuzzy coalitions are studied in [17] where attention is paid to triangular norm-based measures and special extensions of the diagonal Aumann-Shapley value (cf. [4]), and in [8] where the stress in on noncooperative games. Fuzzy and multiobjective games are object of analysis also in [46].

We start this part with the definitions of a fuzzy coalition and a cooperative fuzzy game, and develop the theory of cooperative fuzzy games without paying extensively attention to the ways in which a game with crisp coalitions can be extended to a game with fuzzy coalitions. One possibility is to consider the multilinear extension of a crisp game introduced in [49] or to consider extensions that are given using the Choquet integral (cf. [21]). The reader who is interested in this extension problem is referred to [75] and [76].

This part is organized as follows. Chapter 5 introduces basic notation and notions from cooperative game theory with fuzzy coalitions. In Chapter 6 we have collected various set-valued and one-point solution concepts for fuzzy games like the Aubin core, the dominance core and stable sets, as well as different core catchers and compromise values. Relations among these solution concepts are extensively studied. Chapter 7 is devoted to the notion of convexity of a cooperative fuzzy game. We present several characterizations of a convex fuzzy game and study special properties of solution concepts. For this class of fuzzy games we introduce the notion of a participation monotonic allocation scheme and that of a constrained egalitarian solution. In Chapter 8 we study the cone of fuzzy clan games together with related set-valued solution concepts for these games; the new solution of a bi-monotonic participation allocation scheme is introduced.

5

Preliminaries

Let N be a non-empty set of players usually of the form $\{1, \ldots, n\}$. From now on we systematically refer to elements of 2^N as *crisp coalitions*, and to cooperative games in G^N as *crisp games*.

Definition 5.1. *A **fuzzy coalition** is a vector $s \in [0,1]^N$.*

The i-th coordinate s_i of s is the *participation level* of player i in the fuzzy coalition s. Instead of $[0,1]^N$ we will also write $\mathcal{F}^N$ for the set of fuzzy coalitions on player set N.

A crisp coalition $S \in 2^N$ corresponds in a canonical way with the fuzzy coalition e^S, where $e^S \in \mathcal{F}^N$ is the vector with $\left(e^S\right)^i = 1$ if $i \in S$, and $\left(e^S\right) i = 0$ if $i \in N \setminus S$. The fuzzy coalition e^S corresponds to the situation where the players in S fully cooperate (i.e. with participation levels 1) and the players outside S are not involved at all (i.e. they have participation levels 0). In this part of the book we often refer to fuzzy coalitions e^S with $S \in 2^N$ as *crisp-like coalitions*. We denote by e^i the fuzzy coalition corresponding to the crisp coalition $S = \{i\}$ (and also the i-th standard basis vector in $\mathbb{R}^n$). The fuzzy coalition e^N is called the grand coalition, and the fuzzy coalition (the n-dimensional vector) $e^\emptyset = (0, \ldots, 0)$ corresponds to the empty crisp coalition. We denote the set of all non-empty fuzzy coalitions by $\mathcal{F}_0^N = \mathcal{F}^N \setminus \left\{e^\emptyset\right\}$. Notice that we can identify the fuzzy coalitions with points in the hypercube $[0,1]^N$ and the crisp coalitions with the $2^{|N|}$ extreme points (vertices) of this hypercube. For $N = \{1,2\}$ we have a square with vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. The corresponding geometric picture for $N = \{1,2,3\}$ is that of a cube.

For $s \in \mathcal{F}^N$ we define the carrier of s by $car(s) = \{i \in N \mid s_i > 0\}$ and call s a *proper fuzzy coalition* if $car(s) \neq N$. The set of proper fuzzy coalitions on player set N is denoted by $\mathcal{PF}^N$, and the set of non-empty proper fuzzy coalitions on player set N by $\mathcal{PF}_0^N$.

For $s, t \in \mathcal{F}^N$ we use the notation $s \leq t$ iff $s_i \leq t_i$ for each $i \in N$. We define $s \wedge t = (\min(s_1, t_1), \ldots, \min(s_n, t_n))$ and $s \vee t = (\max(s_1, t_1), \ldots, \max(s_n, t_n))$. The set operations $\vee$ and $\wedge$ play the same role for the fuzzy coalitions as the union and intersection for crisp coalitions.

For $s \in \mathcal{F}^N$ and $t \in [0,1]$, we set $\left(s^{-i} \parallel t\right)$ to be the element in $\mathcal{F}^N$ with $\left(s^{-i} \parallel t\right)_j = s_j$ for each $j \in N \setminus \{i\}$ and $\left(s^{-i} \parallel t\right)_i = t$.

For each $s \in \mathcal{F}^N$ we introduce the *degree of fuzziness* $\varphi(s)$ of s by $\varphi(s) = |\{i \in N \mid s_i \in (0,1)\}|$. Note that $\varphi(s) = 0$ implies that s corresponds to a crisp coalition, and that in a coalition s with $\varphi(s) = n$ no participation level equals 0 or 1.

Definition 5.2. *A* ***cooperative fuzzy game*** *with player set N is a map $v : \mathcal{F}^N \to \mathbb{R}$ with the property $v(e^{\emptyset}) = 0$.*

The map v assigns to each fuzzy coalition a real number, telling what such a coalition can achieve in cooperation.

The set of fuzzy games with player set N will be denoted by FG^N. Note that FG^N is an infinite dimensional linear space.

Example 5.3. Let $v \in FG^{\{1,2,3\}}$ with

$$v(s_1, s_2, s_3) = \min\{s_1 + s_2, s_3\}$$

for each $s = (s_1, s_2, s_3) \in \mathcal{F}^{\{1,2,3\}}$. One can think of a situation where players 1, 2, and 3 have one unit of the infinitely divisible goods A, A, and B, respectively, where A and B are complementary goods, and where combining a fraction α of a unit of A and of B leads to a gain α.

Example 5.4. Let $v \in FG^{\{1,2\}}$ be defined by

$$v(s_1, s_2) = \begin{cases} 1 \text{ if } s_1 \geq \frac{1}{2},\, s_2 \geq \frac{1}{2}, \\ 0 \text{ otherwise,} \end{cases}$$

for each $s = (s_1, s_2) \in \mathcal{F}^{\{1,2\}}$. This game corresponds to a situation is which only coalitions with participation levels of the players of at least $\frac{1}{2}$ are winning, and all other coalitions are losing.

Example 5.5. (A public good game) Suppose n agents want to create a facility for joint use. The cost of the facility depends on the sum of the participation levels of the agents and it is described by $k\left(\sum_{i=1}^n s_i\right)$, where k is a continuous monotonic increasing function on $[0,n]$, with $k(0) = 0$, and where $s_1, \ldots, s_n \in [0,1]$ are the participation levels of the agents. The gain of an agent i with participation level s_i is given by $g_i(s_i)$, where the function $g_i : [0,1] \to \mathbb{R}$ is continuously monotonic increasing with $g_i(0) = 0$. This situation leads to a fuzzy game $v \in FG^N$ where $v(s) = \sum_{i=1}^{n} g_i(s_i) - k\left(\sum_{i=1}^{n} s_i\right)$ for each $s \in F^N$.

For each $s \in \mathcal{F}^N$, let $\lceil s \rfloor := \sum_{i=1}^{n} s_i$ be the *aggregated participation level* of the players in N with respect to s. Given $v \in FG^N$ and $s \in \mathcal{F}_0^N$ we denote by $\alpha(s,v)$ the *average worth* of s with respect to $\lceil s \rfloor$, that is

$$\alpha(s,v) := \frac{v(s)}{\lceil s \rfloor}. \tag{5.1}$$

Note that $\alpha(s,v)$ can be interpreted as a per participation-level-unit value of coalition s.

As it is well known, the notion of a subgame plays an important role in the theory of cooperative crisp games. In what follows, the role of subgames of a crisp game will be taken over by restricted games of a fuzzy game.

For $s,t \in [0,1]^N$ let $s * t$ denote the *coordinate-wise product* of s and t, i.e. $(s*t)_i = s_i t_i$ for all $i \in N$.

Definition 5.6. *Let $v \in FG^N$ and $t \in \mathcal{F}_0^N$. The t-**restricted game** of v is the game $v_t : \mathcal{F}_0^N \to \mathbb{R}$ given by $v_t(s) = v(t*s)$ for all $s \in \mathcal{F}_0^N$.*

In a t-restricted game, $t \in \mathcal{F}_0^N$ plays the role of the grand coalition in the sense that the t-restricted game considers only the subset $\mathcal{F}_t^N$ of $\mathcal{F}_0^N$ consisting of fuzzy coalitions with participation levels of the corresponding players at most t, $\mathcal{F}_t^N = \left\{s \in \mathcal{F}_0^N \mid s \leq t\right\}$.

Remark 5.7. When $t = e^T$ then $v_t(s) = v(e^T * s) = v\left(\sum_{i \in T} s_i e^i\right)$ for each $s \in \mathcal{F}^N$, and for $s = e^S$ we obtain $v_t(e^S) = v(e^{S \cap T})$.

Special attention will be paid to fuzzy unanimity games. In the theory of cooperative crisp games unanimity games play an important role not only because they form a natural basis of the linear space G^N, but also since various interesting classes of games are nicely described with the aid of unanimity games.

For $t \in \mathcal{F}_0^N$, we denote by u_t the fuzzy game defined by

$$u_t(s) = \begin{cases} 1 \text{ if } s \geq t, \\ 0 \text{ otherwise.} \end{cases} \tag{5.2}$$

We call this game the *unanimity game based on t*: a fuzzy coalition s is winning if the participation levels of s exceed weakly the corresponding participation levels of t; otherwise the coalition is losing, i.e. it has value zero.

Remark 5.8. The game in Example 5.4 is the unanimity game with $t = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Of course, the theory of cooperative crisp games has been an inspiration source for the development of the theory of cooperative fuzzy games. In the next chapters we will use operators from FG^N to G^N and from G^N to FG^N (cf. [49], [75], and [76]). In particular, we shall consider the multilinear operator $ml : G^N \to FG^N$ (cf. [49] and (3.9)) and the crisp operator $cr : FG^N \to G^N$. Here for a crisp game $v \in G^N$, *the multilinear extension* $ml(v) \in FG^N$ is defined by

$$ml(v)(s) = \sum_{S \in 2^N \setminus \{\emptyset\}} \left(\prod_{i \in S} s_i \prod_{i \in N \setminus S} (1 - s_i) \right) v(S) \text{ for each } s \in \mathcal{F}^N.$$

For a fuzzy game $v \in FG^N$, the corresponding *crisp game* $cr(v) \in G^N$ is given by

$$cr(v)(S) = v(e^S) \text{ for each } S \in 2^N.$$

Remark 5.9. In view of Remark 5.7, the restriction of $cr(v_{e^T}) : 2^N \to \mathbb{R}$ to 2^T is the subgame of $cr(v)$ on the player set T.

Example 5.10. For the crisp unanimity game u_T the multilinear extension is given by $ml\,(u_T)\,(s) = \prod_{i \in T} s_i$ (cf. [75] and [76]) and $cr\,(ml\,(u_T)) = u_T$. For the games $v, w \in FG^{\{1,2\}}$, where $v\,(s_1, s_2) = s_1\,(s_2)^2$ and $w\,(s_1, s_2) = s_1 \sqrt{s_2}$ for each $s \in \mathcal{F}^{\{1,2\}}$, we have $cr\,(v) = cr\,(w)$.

In general the composition $cr \circ ml : G^N \to G^N$ is the identity map on G^N. But $ml \circ cr : FG^N \to FG^N$ is not the identity map on FG^N if $|N| \geq 2$.

Example 5.11. Let $N = \{1, 2\}$ and let $v \in FG^{\{1,2\}}$ be given by $v(s_1, s_2) = 5 \min \{s_1, 2s_2\}$ for each $s = (s_1, s_2) \in \mathcal{F}^{\{1,2\}}$. Then for the crisp game $cr(v)$ we have $cr(v)(\{1\}) = cr(v)(\{2\}) = 0$ and $cr(v)(\{1, 2\}) = 5$. For the multilinear extension $ml(cr(v))$ we have $ml(cr(v)\,(s)) = 5s_1s_2$. Notice that $ml\left(cr\,(v)\left(1, \frac{1}{2}\right)\right) = 2\frac{1}{2}$ but $v\left(1, \frac{1}{2}\right) = 5$ implying that $v \neq ml(cr(v))$.

Remark 5.12. Note that for a unanimity game u_t, the corresponding crisp game $cr(u_t)$ is equal to u_T, where u_T is the crisp unanimity game based on $T = car(t)$. Conversely, $ml(u_T)$ is for no $T \in 2^N \setminus \{\emptyset\}$ a fuzzy unanimity game because $ml(u_T)$ has a continuum of values: $ml(u_T)\,(s) = \prod_{i \in T} s_i$ for each $s \in \mathcal{F}^N$.

6

Solution concepts for fuzzy games

In this chapter we introduce several solution concepts for fuzzy games and study their properties and interrelations. Sections 6.1-6.3 are devoted to various core concepts and stable sets. The Aubin core introduced in Section 6.1 plays a key role in the rest of this chapter. Section 6.4 presents the Shapley value and the Weber set for fuzzy games which are based on crisp cooperation and serve as an inspiration source for the path solutions and the path solution cover introduced in Section 6.5. Compromise values for fuzzy games are introduced and studied in Section 6.6.

6.1 Imputations and the Aubin core

Let $v \in FG^N$. The *imputation set* $I(v)$ of v is the set

$$\left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(e^N),\ x_i \geq v(e^i) \text{ for each } i \in N \right\}.$$

The Aubin *core* (cf. [1], [2], and [3]) $C(v)$ of a fuzzy game $v \in FG^N$ is the subset of imputations which are stable against any possible deviation by fuzzy coalitions, i.e.

$$C(v) = \left\{ x \in I(v) \mid \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{F}^N \right\}.$$

So $x \in C(v)$ can be seen as a distribution of the value of the grand coalition e^N, where for each fuzzy coalition s, the total payoff is not smaller than $v(s)$, if each player $i \in N$ with participation level s_i is paid $s_i x_i$.

Note that the Aubin core $C(v)$ of $v \in FG^N$ can be also defined as $\left\{x \in I(v) \mid \sum_{i\in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{F}_0^N\right\}$ and this is more in the spirit of the definition of the core of a crisp game (cf. Definition 2.3).

Remark 6.1. The core $C(cr(v))$ of the crisp game corresponding to v includes $C(v) : C(v) \subset C(cr(v))$. We will see in the next chapter that for convex fuzzy games the two cores coincide.

Clearly, the Aubin core $C(v)$ of a fuzzy game v is a closed convex subset of $\mathbb{R}^n$ for each $v \in FG^N$. Of course, the Aubin core may be empty as Example 6.2 shows or can consist of a single element as in Example 6.3.

Example 6.2. Consider again the game v in Example 5.4. The core $C(v)$ is empty because for a core element x it should hold $x_1 + x_2 = v\left(e^{\{1,2\}}\right) = 1$ and also $\frac{1}{2}x_1 + \frac{1}{2}x_2 \geq v\left(\frac{1}{2}, \frac{1}{2}\right) = 1$, which is impossible.

Example 6.3. For the game in Example 5.3, good B is scarce in the grand coalition which is reflected in the fact that the core consists of one point $(0, 0, 1)$, corresponding to the situation where all gains go to player 3 who possesses the scarce good.

The next proposition shows that for a unanimity game u_t (cf. (5.2)) every arbitrary division of 1 among players who have participation level 1 in t generates a core element (cf. [10]).

Proposition 6.4. *Let $u_t \in FG^N$ be the unanimity game based on the fuzzy coalition $t \in \mathcal{F}_0^N$. Then the Aubin core $C(u_t)$ is non-empty iff $t_k = 1$ for some $k \in N$. In fact*

$$C(u_t) = co\left\{e^k \mid k \in N, t_k = 1\right\}.$$

Proof. If $t_k = 1$ for some $k \in N$, then $e^k \in C(u_t)$. Therefore,

$$co\left\{e^k \mid k \in N, t_k = 1\right\} \subset C(u_t).$$

Conversely, $x \in C(u_t)$ implies that $\sum_{i=1}^n x_i = 1 = u_t(e^N)$, $\sum_{i=1}^n t_i x_i \geq 1 = u_t(t)$, $x_i \geq u_t(e^i) \geq 0$ for each $i \in N$. So $x \geq 0$, $\sum_{i=1}^n x_i(1 - t_i) \leq 0$, which implies that $x_i(1 - t_i) = 0$ for all $i \in N$. Hence, $\{x_i \mid x_i > 0\} \subset \{i \in N \mid t_i = 1\}$ and, consequently, $x \in co\left\{e^k \mid k \in N, t_k = 1\right\}$. So,

$$C(u_t) \subset co\left\{e^k \mid k \in N, t_k = 1\right\}.$$

In what follows, we denote by FG_*^N the set of fuzzy games with a non-empty (Aubin) core.

6.2 Other cores and stable sets

Now we introduce two other cores for a fuzzy game $v \in FG^N$, namely the proper core and the crisp core, by weakening the stability conditions in the definition of the Aubin core (cf. [70]).

To define the *proper core* $C^P(v)$ of a fuzzy game v we consider only stability regarding proper fuzzy coalitions (cf. Chapter 5, page 50), i.e.

$$C^P(v) = \left\{ x \in I(v) \mid \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{PF}^N \right\}.$$

Note that $C^P(v)$ can be also defined as

$$\left\{ x \in I(v) \mid \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{PF}_0^N \right\}.$$

Further, if we consider only crisp-like coalitions e^S in the stability conditions, one obtains the *crisp core* $C^{cr}(v)$ of the fuzzy game $v \in FG^N$, i.e.

$$C^{cr}(v) = \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(e^S) \text{ for each } S \in 2^N \right\}.$$

Clearly, the crisp core $C^{cr}(v)$ of a fuzzy game v can be also defined as $\left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(e^S) \text{ for each } S \in 2^N \setminus \{\emptyset\} \right\}$ and it is also the core of the crisp game $cr(v)$. One can easily see that both cores $C^P(v)$ and $C^{cr}(v)$ are convex sets.

Let $v \in FG^N$, $x, y \in I(v)$ and let $s \in \mathcal{F}_0^N$. We say that x *dominates* y *via* s, denoted by $x \operatorname{dom}_s y$, if

(i) $x_i > y_i$ for all $i \in car(s)$, and

(ii) $\sum_{i \in N} s_i x_i \leq v(s)$.

The two above conditions are interpreted as follows:

- $x_i > y_i$ implies $s_i x_i > s_i y_i$ for each $i \in car(s)$, which means that the imputation $x = (x_1, ..., x_n)$ is better than the imputation $y = (y_1, ..., y_n)$ for all (active) players $i \in car(s)$;
- $\sum_{i \in N} s_i x_i \leq v(s)$ means that the payoff $\sum_{i \in N} s_i x_i$ is reachable by the fuzzy coalition s.

Remark 6.5. Note that $x \operatorname{dom}_s y$ implies $s \in \mathcal{PF}_0^N$ because from $x_i > y_i$ for all $i \in N$ it follows $\sum_{i \in N} x_i > \sum_{i \in N} y_i$, in contradiction with $x, y \in I(v)$. It is, however, to be noted that $|car(s)| = 1$ is possible.

We simply say x *dominates* y, denoted by $x \operatorname{dom} y$, if there is a non-empty (proper) fuzzy coalition s such that $x \operatorname{dom}_s y$. The negation of $x \operatorname{dom} y$ is denoted by $\neg\, x \operatorname{dom} y$.

Definition 6.6. *The* ***dominance core*** $DC(v)$ *of a fuzzy game* $v \in FG^N$ *is the set of imputations which are not dominated by any other imputation,*

$$DC(v) = \{x \in I(v) \mid \neg\, y \operatorname{dom} x \text{ for all } y \in I(v)\}.$$

Definition 6.7. *A* ***stable set*** *of a fuzzy game* $v \in FG^N$ *is a nonempty set* K *of imputations satisfying the properties:*
(i) (Internal stability) For all $x, y \in K$, $\neg\, x \operatorname{dom} y$;
(ii) (External stability) For all $z \in I(v) \setminus K$, *there is an imputation* $x \in K$ *such that* $x \operatorname{dom} z$.

Theorem 6.8. *Let* $v \in FG^N$. *Then*
(i) $C(v) \subset C^P(v) \subset C^{cr}(v)$;
(ii) $C^P(v) \subset DC(v)$;
(iii) for each stable set K *it holds that* $DC(v) \subset K$.

Proof. The theorem is trivially true if $I(v) = \emptyset$. So, suppose in the following that $I(v) \neq \emptyset$.

(i) This follows straightforwardly from the definitions.

(ii) Let $x \in I(v) \setminus DC(v)$. Then there are $y \in I(v)$ and $s \in \mathcal{PF}_0^N$ satisfying $y_i > x_i$ for each $i \in car(s)$ and $\sum_{i \in N} s_i y_i \leq v(s)$. Then $\sum_{i \in car(s)} s_i x_i < \sum_{i \in car(s)} s_i y_i \leq v(s)$. Hence $x \in I(v) \setminus C^P(v)$. We conclude that $C^P(v) \subset DC(v)$.

(iii) Let K be a stable set. Since $DC(v)$ consists of undominated imputations and each imputation in $I(v) \setminus K$ is dominated by some imputation by the external stability property, it follows that $DC(v) \subset K$.

In the next theorem we give sufficient conditions for the coincidence of the proper core and the dominance core for fuzzy games.

Theorem 6.9. *Let* $v \in FG^N$. *Suppose* $v\left(e^N\right) - \sum_{i \in N \setminus car(s)} v\left(e^i\right) - \frac{v(s)}{s^*} \geq 0$ *for each* $s \in \mathcal{F}_0^N$, *where* $s^* = \min_{i \in car(s)} s_i$. *Then* $C^P(v) = DC(v)$.

Proof. Note that $C^P(v) = DC(v) = \emptyset$ if $I(v) = \emptyset$. Suppose $I(v) \neq \emptyset$. From Theorem 6.8 it follows that $C^P(v) \subset DC(v)$. We show the converse inclusion by proving that $x \notin C^P(v)$ implies $x \notin DC(v)$. Let $x \in I(v) \setminus C^P(v)$. Then there is $s \in \mathcal{PF}_0^N$ such that $\sum_{i \in N} s_i x_i < v(s)$. For each $i \in car(s)$ take $\varepsilon_i > 0$ such that $\sum_{i \in car(s)} s_i (x_i + \varepsilon_i) = v(s)$. Since

$$\sum_{i \in car(s)} (x_i + \varepsilon_i) \leq \sum_{i \in car(s)} \frac{s_i (x_i + \varepsilon_i)}{s^*} = \frac{v(s)}{s^*},$$

we can take $\delta_i \geq 0$ for each $i \notin car(s)$ such that

$$\sum_{i \notin car(s)} \delta_i = \frac{v(s)}{s^*} - \sum_{i \in car(s)} (x_i + \varepsilon_i).$$

Further, we define $y \in \mathbb{R}^N$ by

$$y_i = \begin{cases} x_i + \varepsilon_i & \text{for each } i \in car(s), \\ v\left(e^i\right) + \frac{v(e^N) - \sum_{i \in N \setminus car(s)} v(e^i) - \frac{v(s)}{s^*}}{|N \setminus car(s)|} + \delta_i & \text{for each } i \notin car(s). \end{cases}$$

Note that $\sum_{i \in N} y_i = v(e^N)$, $y_i > x_i > v(e^i)$ for each $i \in car(s)$ and, since $v(e^N) - \sum_{i \in N \setminus car(s)} v(e^i) - \frac{v(s)}{s^*} \geq 0$, we have $y_i \geq v(e^i)$ for each $i \in N \setminus car(s)$. Hence $y \in I(v)$. Now, since $y_i > x_i$ for all $i \in car(s)$ and $\sum_{i \in N} s_i y_i = v(s)$ we have $y \operatorname{dom}_s x$; thus $x \in I(v) \setminus DC(v)$.

Remark 6.10. Let $v \in FG^N$. Take the crisp game $w = cr(v)$. Then $v(e^N) \geq \frac{v(s)}{s^*} + \sum_{i \in N \setminus car(s)} v(e^i)$ for each $s \in \mathcal{F}_0^N$ implies $w(N) \geq w(S) + \sum_{i \in N \setminus S} w(i)$, for each $S \subseteq N$. So, Theorem 6.9 can be seen as an extension of the corresponding property for cooperative crisp games (cf. Theorem 2.12(i)).

From Theorem 6.9 we obtain the following corollary.

Corollary 6.11. *Let $v \in FG^N$ with $v(e^i) \geq 0$ for each $i \in N$ and $C^P(v) \neq DC(v)$. Then $C^P(v) = \emptyset$.*

Proof. $C^P(v) \neq DC(v)$ implies that $I(v) \neq \emptyset$ and that there is $t \in \mathcal{PF}^N$ with $v(t) + \sum_{i \in N \setminus car(t)} v(e^i) > v(e^N)$ by Theorem 6.9. By $v(e^i) \geq 0$ for each $i \in N$, we have $x \geq 0$ and $\sum_{i \in N} x_i = v(e^N)$ for each $x \in I(v)$. Hence

$$\begin{aligned} \sum_{i \in N} t_i x_i &= \sum_{i \in car(t)} t_i x_i \\ &\leq \sum_{i \in car(t)} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus car(t)} x_i \\ &\leq v(e^N) - \sum_{i \in N \setminus car(t)} v(e^i) \\ &< v(t) \end{aligned}$$

holds for each $x \in I(v)$. Thus there is no $x \in I(v)$ such that $x \in C^P(v)$. Hence, $C^P(v) = \emptyset$.

Next we prove that for a fuzzy game the dominance core is a convex set.

Lemma 6.12. *Let $v \in FG^N$ with $v(e^i) \geq 0$ for each $i \in N$. Let $\bar{v} \in FG^N$ be given by*

$$\bar{v}(s) = \min \left\{ v(s), v(e^N) - \sum_{i \in N \setminus car(s)} v(e^i) \right\}.$$

Then $DC(v) = DC(\bar{v}) = C^P(\bar{v})$.

Proof. Note that $DC(v) = DC(\bar{v}) = C^P(\bar{v}) = \emptyset$ if $I(v) = \emptyset$. Suppose $I(v) \neq \emptyset$. It implies $I(v) = I(\bar{v})$. Thus, to prove $DC(v) = DC(\bar{v})$, it is sufficient to show that for $x, y \in I(v)$ and $s \in \mathcal{F}_0^N$, $x \operatorname{dom}_s y$ in v if and only if $x \operatorname{dom}_s y$ in $\bar{v}$. We only have to show that for $x \in I(v)$ and $s \in \mathcal{F}_0^N$, $\sum_{i\in N} s_i x_i \leq v(s)$ if and only if $\sum_{i\in N} s_i x_i \leq \bar{v}(s)$.

The 'if' part follows from $\bar{v}(s) \leq v(s)$. For the 'only if' part note that for $s \in \mathcal{F}_0^N$ and $x \in I(v)$ we have $\sum_{i\in N} s_i x_i = \sum_{i\in car(s)} s_i x_i \leq \sum_{i\in car(s)} x_i = \sum_{i\in N} x_i - \sum_{i\in N\setminus car(s)} x_i \leq v(e^N) - \sum_{i\in N\setminus car(s)} v(e^i)$, where the first inequality follows from $x_i \geq v(e^i) \geq 0$ for each $i \in car(s) \subseteq N$. Hence, $\sum_{i\in N} s_i x_i \leq v(s)$ implies $\sum_{i\in N} s_i x_i \leq \bar{v}(s)$. Since we have $\bar{v}(s) + \sum_{i\in N\setminus car(s)} \bar{v}(e^i) \leq \bar{v}(s) + \sum_{i\in N\setminus car(s)} v(e^i) \leq v(e^N) = \bar{v}(e^N)$ by $\bar{v}(e^i) \leq v(e^i)$, we obtain $DC(\bar{v}) = C^P(\bar{v})$ by Theorem 6.9.

Theorem 6.13. *For each $v \in FG^N$, $DC(v)$ is a convex set.*

Proof. Let $v \in FG^N$. Define the fuzzy game v' by $v'(s) = v(s) - \sum_{i\in car(s)} s_i v(e^i)$ for each $s \in \mathcal{F}^N$. Note that $v'(e^i) = 0$ for each $i \in N$. From Lemma 6.12 it follows that $DC(v')$ is a convex set, because $C^P(\bar{v}')$ is a convex set. Now, we use the fact that for an arbitrary fuzzy game v, $DC(v) = DC(v') + (v(e^1), \ldots, v(e^n))$ holds, where $DC(v') + (v(e^1), \ldots, v(e^n)) = \{x + y \mid x \in DC(v'),\ y = (v(e^1), \ldots, v(e^n))\}$.

Now we give two examples to illustrate the results in the above theorems.

Example 6.14. Let $N = \{1, 2\}$ and let $v : \mathcal{F}^{\{1,2\}} \to \mathbb{R}$ be given by $v(s_1, s_2) = s_1 + s_2 - 1$ for each $s \in \mathcal{F}_0^{\{1,2\}}$ and $v(e^\emptyset) = 0$. Further, let

$$v_1(s) = \begin{cases} v(s) & \text{if } s \neq \left(0, \frac{1}{2}\right), \\ 4 & \text{if } s = \left(0, \frac{1}{2}\right). \end{cases}$$

$$v_2(s) = \begin{cases} v(s) & \text{if } s \neq \left(\frac{1}{2}, \frac{1}{2}\right), \\ 4 & \text{if } s = \left(\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

Let $\Delta = \{x \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 = 1\}$. Then
(i) $C(v) = C^P(v) = DC(v) = I(v) = \Delta$,
(ii) $C(v_1) = C^P(v_1) = \emptyset,\ DC(v_1) = I(v_1) = \Delta$,
(iii) $C(v_2) = \emptyset,\ C^P(v_2) = DC(v_2) = I(v_2) = \Delta$,
(iv) for v, v_1, and v_2, the imputation set Δ is the unique stable set.
Note that $v_2(s) + \sum_{i\in N\setminus car(s)} v_2(e^i) > v_2(e^N)$ for $s = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $C^P(v_2) = DC(v_2)$. Hence, the sufficient condition in Theorem 6.9 for the equality $C^P(v) = DC(v)$ is not a necessary condition.

In the next example we give a fuzzy game v with $C(v) \neq DC(v)$ and $C(v) \neq \emptyset$. Notice that for a crisp game w we have that $C(w) = \emptyset$ if $C(w) \neq DC(w)$ (cf. Theorem 2.12(ii)).

Example 6.15. Let $N = \{1,2\}$ and let $v : \mathcal{F}^{\{1,2\}} \to \mathbb{R}$ be given by $v(s_1, 1) = \sqrt{s_1}$ for all $(s_1, 1) \in \mathcal{F}^{\{1,2\}}$, and $v(s_1, s_2) = 0$ otherwise. Then $I(v) = \left\{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 + x_2 = 1\right\}$, $C(v) = \left\{x \in I(v) \mid 0 \leq x_1 \leq \frac{1}{2}\right\} \neq I(v)$, and $C^P(v) = DC(v) = I(v)$. Further, $I(v)$ is the unique stable set.

By using the average worth of a coalition $s \in \mathcal{F}_0^N$ in a game $v \in FG^N$ (cf. (5.1)), we define the *equal division core* $EDC(v)$ of the game v as the set

$$\left\{x \in I(v) \mid \nexists s \in \mathcal{F}_0^N \text{ s.t. } \alpha\,(s, v) > x_i \text{ for all } i \in car(s)\right\}.$$

So $x \in EDC(v)$ can be seen as a distribution of the value of the grand coalition e^N, where for each fuzzy coalition s, there is a player i with a positive participation level for which the payoff x_i is at least as good as the equal division share $\alpha\,(s, v)$ of $v(s)$ in s.

Proposition 6.16. *Let $v \in FG^N$. Then $EDC(v) \subset EDC(cr(v))$.*

Proof. Suppose $x \in EDC\,(v)$. Then by the definition of $EDC(v)$ there is no $e^S \in \mathcal{F}_0^N$ s.t. $\alpha\left(e^S, v\right) > x_i$ for all $i \in car(e^S)$. Taking into account that $cr(v)\,(S) = v\left(e^S\right)$ for all $S \in 2^N$, there is no $S \neq \emptyset$ s.t. $\frac{cr(v)(S)}{|S|} > x_i$ for all $i \in S$. Hence, $x \in EDC\,(cr(v))$.

The next example shows that $EDC(v)$ and $EDC(cr(v))$ are not necessarily equal.

Example 6.17. Let $N = \{1, 2, 3\}$ and $v(s_1, s_2, s_3) = \sqrt{s_1 + s_2 + s_3}$ for each $s = (s_1, s_2, s_3) \in \mathcal{F}^{\{1,2,3\}}$. For this game we have

$$EDC(cr(v)) = \left\{\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right\} \text{ and } EDC(v) = \emptyset.$$

Remark 6.18. We refer the reader to [36] for an analysis of other core-like solution concepts for cooperative fuzzy games.

6.3 The Shapley value and the Weber set

Let $\pi(N)$ be the set of linear orderings of N. We introduce for $v \in FG^N$ the *marginal vectors* $m^\sigma(v)$ for each $\sigma \in \pi(N)$, the *fuzzy Shapley value* $\phi(v)$ and the *fuzzy Weber set* $W(v)$ as follows (cf. [10]):

(*i*) $m^\sigma(v) = m^\sigma(cr(v))$ for each $\sigma \in \pi(N)$;

(*ii*) $\phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \pi(N)} m^\sigma(v)$;

(*iii*) $W(v) = conv\,\{m^\sigma(v) \mid \sigma \in \pi(N)\}$.

Note that $\phi(v) = \phi(cr(v))$, $W(v) = W(cr(v))$. Note further that for $i = \sigma(k)$, the i-th coordinate $m_i^\sigma(v)$ of the marginal vector $m^\sigma(v)$ is given by

$$m_i^\sigma(v) = v\left(\sum_{r=1}^{k} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right).$$

One can identify each $\sigma \in \pi(N)$ with an n-step walk along the edges of the hypercube of fuzzy coalitions starting in $e^{\emptyset}$ and ending in e^N by passing the vertices $e^{\sigma(1)}, e^{\sigma(1)} + e^{\sigma(2)}, \ldots, \sum_{r=1}^{n-1} e^{\sigma(r)}$. The vector $m^\sigma(v)$ records the changes in value from vertex to vertex. The result in [74] that the core of a crisp game is a subset of the Weber set of the game can be extended for fuzzy games as we see in

Proposition 6.19. *Let $v \in FG^N$. Then $C(v) \subset W(v)$.*

Proof. By Remark 6.1 we have $C(v) \subset C(cr(v))$ and by Theorem 2.18, $C(cr(v)) \subset W(cr(v))$. Since $W(cr(v)) = W(v)$ we obtain $C(v) \subset W(v)$.

Note that the Weber set and the fuzzy Shapley value of a fuzzy game are very robust solution concepts since they are completely determined by the possibilities of crisp cooperation, regardless of what are the extra options that players could have as a result of graduating their participation rates. More specific solution concepts for fuzzy games will be introduced in the next sections of this chapter.

Inspired by [49] one can define the *diagonal value* $\delta(v)$ for a C^1-fuzzy game v (i.e. a game whose characteristic function is differentiable with continuous derivatives) as follows: for each $i \in N$ the i-th coordinate $\delta_i(v)$ of $\delta(v)$ is given by

$$\delta_i(v) = \int_0^1 D_i v(t, t, \ldots, t)dt,$$

where D_i is the partial derivative of v with respect to the i-th coordinate. According to Theorem 3.9 we have that for each crisp game $v \in G^N$:

$$\phi_i(v) = \delta_i(ml(v)) \text{ for each } i \in N.$$

The next example shows that for a fuzzy game v, $\delta(v)$ and $\phi(cr(v))$ may differ.

Example 6.20. Let $v \in FG^{\{1,2\}}$ with $v(s_1, s_2) = s_1(s_2)^2$ for each $s = (s_1, s_2) \in \mathcal{F}^{\{1,2\}}$. Then

$$m^{(1,2)} = (v(1,0) - v(0,0), v(1,1) - v(1,0)) = (0,1),$$

$$m^{(2,1)} = (v(1,1) - v(0,1), v(0,1) - v(0,0)) = (1,0);$$

so,

$$\phi(v) = \frac{1}{2}((0,1) + (1,0)) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Further,

$$D_1 v(s_1, s_2) = (s_2)^2,\ D_2 v(s_1, s_2) = 2s_1 s_2;$$

so,

$$\delta_1(v) = \int_0^1 t^2 dt = \frac{1}{3},\ \delta_2(v) = \int_0^1 2t^2 dt = \frac{2}{3}.$$

Hence,

$$\delta(v) = \left(\frac{1}{3}, \frac{2}{3}\right) \neq \left(\frac{1}{2}, \frac{1}{2}\right) = \phi(v).$$

The diagonal value is in fact the fuzzy value studied in [2], [3]. For extensions of this value the reader is referred to [17].

6.4 Path solutions and the path solution cover

Let us consider paths in the hypercube $[0,1]^N$ of fuzzy coalitions, which connect $e^{\emptyset}$ with e^N in a special way (cf. [12]).

Formally, a sequence $q = \langle p^0, p^1, \ldots, p^m \rangle$ of $m+1$ different points in $\mathcal{F}^N$ will be called a *path* (of length m) in $[0,1]^N$ if

(i) $p^0 = (0,0,\ldots,0)$, and $p^m = (1,1,\ldots,1)$;

(ii) $p^k \leq p^{k+1}$ for each $k \in \{0,\ldots,m-1\}$;

(iii) for each $k \in \{0,\ldots,m-1\}$, there is one player $i \in N$ (the acting player in point p^k) such that $\left(p^k\right)_j = \left(p^{k+1}\right)_j$ for all $j \in N \setminus \{i\}$, $\left(p^k\right)_i < \left(p^{k+1}\right)_i$.

For a path $q = \langle p^0, p^1, \ldots, p^m \rangle$ let us denote by $Q_i(q)$ the set of points p^k, where player i is acting, i.e. where $\left(p^k\right)_i < \left(p^{k+1}\right)_i$. Given a game $v \in FG^N$ and a path q, the payoff vector $x^q(v) \in \mathbb{R}^n$ corresponding to v and q has the i-th coordinate

$$x_i^q(v) = \sum_{k:p^k \in Q_i(q)} \left(v(p^{k+1}) - v(p^k)\right),$$

for each $i \in N$.

Given such a path $\langle p^0, p^1, \ldots, p^m \rangle$ of length m and $v \in FG^N$, one can imagine the situation, where the players in N, starting from non-cooperation $\left(p^0 = 0\right)$ arrive to full cooperation $\left(p^m = e^N\right)$ in m steps, where in each step one of the players increases his participation level. If the increase in value in such a step is given to the acting player, the resulting aggregate payoffs lead to the vector $x^q(v) = (x_i^q(v))_{i \in N}$. Note that $x^q(v)$ is an efficient vector, i.e. $\sum_{i=1}^n x_i^q(v) = v\left(e^N\right)$. We call $x^q(v)$ a *path solution*.

Let us denote by $Q(N)$ the set of paths in $[0,1]^N$. Then we denote by $Q(v)$ the convex hull of the set of path solutions and call it the *path solution cover*. Hence,

$$Q(v) = co\left\{x^q(v) \in \mathbb{R}^n \mid q \in Q(N)\right\}.$$

Note that all paths $q \in Q(N)$ have length at least n. There are $n!$ paths with length exactly n; each of these paths corresponds to a situation where one by one the players $-$ say in the order $\sigma(1), \ldots, \sigma(n)$ $-$ increase their participation from level 0 to level 1. Let us denote such a path along n edges by q^{σ}. Then

$$q^{\sigma} = \left\langle 0, e^{\sigma(1)}, e^{\sigma(1)} + e^{\sigma(2)}, \ldots, e^{N} \right\rangle.$$

Clearly, $x\left(q^{\sigma}\right) = m^{\sigma}(v)$. Hence,

$$W(v) = co\left\{x\left(q^{\sigma}\right) \mid \sigma \text{ is an ordering of } N\right\} \subset Q(v).$$

According to Proposition 6.19, the core of a fuzzy game is a subset of the Weber set. Hence

Proposition 6.21. *For each $v \in FG^{N}$ we have $C(v) \subset W(v) \subset Q(v)$.*

Example 6.22. Let $v \in FG^{\{1,2\}}$ be given by $v(s_1, s_2) = s_1\left(s_2\right)^2 + s_1 + 2s_2$ for each $s = (s_1, s_2) \in \mathcal{F}^{\{1,2\}}$ and let $q \in Q(N)$ be the path of length 3 given by $\left\langle (0,0), \left(\frac{1}{3}, 0\right), \left(\frac{1}{3}, 1\right), (1,1)\right\rangle$. Then $x_1^q(v) = \left(v\left(\frac{1}{3}, 0\right) - v(0,0)\right) + \left(v(1,1) - v\left(\frac{1}{3}, 1\right)\right) = 1\frac{2}{3}$, $x_2^q(v) = v\left(\frac{1}{3}, 1\right) - v\left(\frac{1}{3}, 0\right) = 2\frac{1}{3}$. So $\left(1\frac{2}{3}, 2\frac{1}{3}\right) \in Q(v)$. The two shortest paths of length 2 given by $q^{(1,2)} = \langle (0,0), (1,0), (1,1)\rangle$ and $q^{(2,1)} = \langle (0,0), (0,1), (1,1)\rangle$ have payoff vectors $m^{(1,2)}(v) = (1,3)$, and $m^{(2,1)}(v) = (2,2)$, respectively.

Keeping in mind the interrelations among the Aubin core, the fuzzy Weber set and the path solution cover, one can try to introduce lower and upper bounds for payoff vectors in these sets. A lower (upper) bound is a payoff vector whose i-th coordinate is at most (at least) as good as the payoff given to player i when a "least desirable" ("most convenient") situation for him is achieved. By using pairs consisting of a lower bound and an upper bound, we obtain hypercubes which are catchers of the Aubin core, the fuzzy Weber set, and the path solution cover, respectively. In the next section we obtain compromise values for fuzzy games by taking a feasible compromise between the lower and upper bounds of the three catchers.

Formally, a *hypercube* in $\mathbb{R}^n$ is a set of vectors of the form

$$[a,b] = \left\{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for each } i \in N\right\},$$

where $a, b \in \mathbb{R}^n$, $a \leq b$ (and the order $\leq$ is the standard partial order in $\mathbb{R}^n$). The vectors a and b are called *bounding vectors* of the hypercube $[a,b]$, where, more explicitly, a is called the *lower vector* and b the *upper vector* of $[a,b]$. Given a set $A \subset \mathbb{R}^n$ we say that the hypercube $[a,b]$ is a *catcher* of A if $A \subset [a,b]$, and $[a,b]$ is called a *tight catcher* of A if there is no hypercube strictly included in $[a,b]$ which also catches A.

A hypercube of reasonable outcomes for a cooperative crisp game plays a role in [42] (see also [29]) and this hypercube can be seen as a tight catcher of the Weber set for crisp games (cf. Section 2.2). Also in [64] and [68] hypercubes are considered which are catchers of the core of crisp games.

Our aim is to introduce and study catchers of the Aubin core, the fuzzy Weber set and the path solution cover for games with a non-empty Aubin core, i.e. games which belong to FG_*^N.

Let us first introduce a core catcher

$$HC(v) = [l\,(C(v))\,, u\,(C(v))]$$

for a game $v \in FG_*^N$, where for each $k \in N$:

$$l_k\,(C\,(v)) = \sup \left\{\varepsilon^{-1} v\left(\varepsilon e^k\right) \mid \varepsilon \in (0,1]\right\},$$

and

$$u_k\,(C\,(v)) = \inf \left\{\varepsilon^{-1}\left(v(e^N) - v\left(e^N - \varepsilon e^k\right)\right) \mid \varepsilon \in (0,1]\right\}.$$

Proposition 6.23. *For each $v \in FG_*^N$ and each $k \in N$:*

$$-\infty < l_k\,(C(v)) \le u_k\,(C(v)) < \infty \text{ and } C(v) \subset HC(v).$$

Proof. Let $x \in C(v)$.

(i) For each $k \in N$ and $\varepsilon \in (0,1]$ we have

$$v\left(e^N\right) - v\left(e^N - \varepsilon e^k\right) \ge \sum_{i \in N} x_i - \left((1-\varepsilon)\,x_k + \sum_{i \in N \setminus \{k\}} x_i\right) = \varepsilon x_k.$$

So,

$$x_k \le \varepsilon^{-1}\left(v(e^N) - v\left(e^N - \varepsilon e^k\right)\right)$$

implying that

$$x_k \le u_k\,(C(v)) < \infty.$$

(ii) For each $\varepsilon \in (0,1]$ we have $\varepsilon x_k \ge v\left(\varepsilon e^k\right)$. Hence,

$$x_k \ge \sup\left\{\varepsilon^{-1} v\left(\varepsilon e^k\right) \mid \varepsilon \in (0,1]\right\} = l_k\,(C\,(v)) > -\infty.$$

By using (i) and (ii) one obtains the inequalities in the proposition and the fact that $HC(v)$ is a catcher of $C(v)$.

Now we introduce for each $v \in FG_*^N$ a fuzzy variant $HW(v)$ of the hypercube of reasonable outcomes introduced in [42],

$$HW(v) = [l\,(W(v))\,, u\,(W(v))]\,,$$

where for each $k \in N$:

$$l_k\left(W(v)\right)=\min\left\{v\left(e^{S\cup\{k\}}\right)-v\left(e^{S}\right)\mid S\subset N\setminus\{k\}\right\},$$

and

$$u_k\left(W(v)\right)=\max\left\{v\left(e^{S\cup\{k\}}\right)-v\left(e^{S}\right)\mid S\subset N\setminus\{k\}\right\}.$$

Then we have

Proposition 6.24. *For each $v\in FG_*^N$ the hypercube $HW(v)$ is a tight catcher of $W(v)$.*

Proof. Left to the reader.

Let us call a set $[a,b]$ with $a\le b$, $a\in(\mathbb{R}\cup\{-\infty\})^n$ and $b\in(\mathbb{R}\cup\{\infty\})^n$ a *generalized* hypercube.

Now we introduce for $v\in FG_*^N$ the generalized hypercube

$$HQ(v)=\left[l\left(Q(v)\right),u\left(Q(v)\right)\right],$$

which catches the path solution cover $Q(v)$ as we see in Theorem 6.25, where for $k\in N$:

$$l_k\left(Q(v)\right)=\inf\left\{\varepsilon^{-1}\left(v\left(s+\varepsilon e^k\right)-v(s)\right)\mid s\in\mathcal{F}^N, s_k<1, \varepsilon\in(0,1-s_k]\right\},$$

$$u_k\left(Q(v)\right)=\sup\left\{\varepsilon^{-1}\left(v\left(s+\varepsilon e^k\right)-v(s)\right)\mid s\in\mathcal{F}^N, s_k<1, \varepsilon\in(0,1-s_k]\right\},$$

where $l_k\left(Q(v)\right)\in[-\infty,\infty)$ and $u_k\left(Q(v)\right)\in(-\infty,\infty]$.

Note that $u\left(Q(v)\right)\ge u\left(C(v)\right)$, $l\left(Q(v)\right)\le l\left(C(v)\right)$.

Theorem 6.25. *For $v\in FG_*^N$, $HQ(v)$ is a catcher of $Q(v)$.*

Proof. This assertion follows from the fact that for each path $q\in Q(N)$ and any $i\in N$

$$\begin{aligned} x_i^q(v) &= \sum_{k:p^k\in Q_i(q)}\left(v\left(p^k+\left(p_i^{k+1}-p_i^k\right)e^i\right)-v\left(p^k\right)\right)\\ &\le \sum_{k:p^k\in Q_i(q)}\left(p_i^{k+1}-p_i^k\right)u_i\left(Q(v)\right)\\ &= u_i\left(Q(v)\right),\end{aligned}$$

and, similarly,

$$x_i^q(v)\ge l_i\left(Q(v)\right).$$

Note that the lower and upper bounds of the catcher of the fuzzy Weber set are obtained by using a finite number of value differences, where only coalitions corresponding to crisp coalitions play a role. The calculation of the lower and upper bounds of the catchers of the Aubin core and of the path solution cover is based on infinite value differences.

6.5 Compromise values

We introduce now for fuzzy games compromise values of σ-type and of τ-type with respect to each of the set-valued solutions C, W and Q. For the first type use is made directly of the bounding vectors of $HC(v), HW(v)$ and $HQ(v)$, while for the τ-type compromise values the upper vector is used together with the remainder vector derived from the upper vector (cf. [12]).

To start with the first type, consider a hypercube $[a,b]$ in $\mathbb{R}^n$ and a game $v \in FG_*^N$ such that the hypercube contains at least one efficient vector, i.e.

$$[a,b] \cap \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v\left(e^N\right) \right\} \neq \emptyset.$$

Then there is a unique point $c\,(a,b)$ on the line through a and b which is also efficient in the sense that $\sum_{i=1}^{n} c_i\,(a,b) = v\left(e^N\right)$. So $c\,(a,b)$ is the convex combination of a and b, which is efficient. We call $c(a,b)$ the *feasible compromise* between a and b.

Now we introduce the following three σ-like compromises for $v \in FG_*^N$:

$$val_C^{\sigma}(v) = c\,(HC(v)) = c\,([l\,(C(v))\,, u\,(C(v))])\,,$$

$$val_W^{\sigma}(v) = c\,(HW(v)) = c\,([l\,(W(v))\,, u\,(W(v))])\,,$$

and

$$val_Q^{\sigma}(v) = c\,(HQ(v)) = c\,([l\,(Q(v))\,, u\,(Q(v))])$$

if the generalized hypercube $HQ(v)$ is a hypercube.

Note that

$$\emptyset \neq C(v) \subset HC(v) \subset HQ(v), \tag{6.1}$$

and

$$\emptyset \neq C(v) \subset W(v) \subset HW(v), \tag{6.2}$$

so all hypercubes contain efficient vectors and the first two compromise value vectors are always well defined.

For the τ-like compromise values inspired by the definition of minimal right vectors for crisp games (cf. [5], [24], and Section 2.2) we define the fuzzy minimal right operator $m^v : \mathbb{R}^n \to \mathbb{R}^n$ for $v \in FG_*^N$ by

$$m_i^v(z) = \sup \left\{ s_i^{-1} \left(v(s) - \sum_{j \in N \setminus \{i\}} s_j z_j \right) \mid s \in \mathcal{F}_0^N, s_i > 0 \right\}$$

for each $i \in N$ and each $z \in \mathbb{R}^n$.

The following proposition shows that m^v assigns to each upper bound z of the Aubin core (i.e. $z \geq x$ for each $x \in C(v)$) a lower bound $m^v(z)$ of $C(v)$, called the *remainder vector* corresponding to z.

Proposition 6.26. *Let $v \in FG_*^N$ and let $z \in \mathbb{R}^n$ be an upper bound of $C(v)$. Then $m^v(z)$ is a lower bound of $C(v)$.*

Proof. Let $i \in N$ and $x \in C(v)$. For each $s \in \mathcal{F}^N$ with $s_i > 0$ we have

$$\begin{aligned} s_i^{-1}\left(v(s) - \sum_{j \in N\setminus\{i\}} s_j z_j\right) &\leq s_i^{-1}\left(\sum_{j \in N} s_j x_j - \sum_{j \in N\setminus\{i\}} s_j z_j\right) \\ &= x_i + s_i^{-1} \sum_{j \in N\setminus\{i\}} s_j\left(x_j - z_j\right) \\ &\leq x_i, \end{aligned}$$

where the first inequality follows from $x \in C(v)$ and the second inequality from the fact that z is an upper bound for $C(v)$, and then $z \geq x$. Hence $m_i^v(z) \leq x_i$ for each $i \in N$, which means that $m^v(z)$ is a lower bound for $C(v)$.

Now we are able to introduce the τ-like compromise values taking into account that all upper vectors of $HC(v), HW(v)$ and $HQ(v)$ are upper bounds for the Aubin core of $v \in FG_*^N$ as follows from (6.1) and (6.2).

So the following definitions make sense for $v \in FG_*^N$:

$$val_C^\tau(v) = c\left(\left[m^v\left(u\left(C(v)\right)\right), u\left(C(v)\right)\right]\right),$$

$$val_W^\tau(v) = c\left(\left[m^v\left(u\left(W(v)\right)\right), u\left(W(v)\right)\right]\right),$$

and

$$val_Q^\tau(v) = c\left(\left[m^v\left(u\left(Q(v)\right)\right), u\left(Q(v)\right)\right]\right)$$

if the generalized hypercube $HQ(v)$ is a hypercube.

The compromise value $val_C^\tau(v)$ is in the spirit of the τ-value introduced in [64] for cooperative crisp games (see Section 3.2), and the compromise value $val_W^\tau(v)$ is in the spirit of the χ-value in [6], the μ-value in [33] and one of the values in [15] and [16] for cooperative crisp games.

7

Convex fuzzy games

An interesting class of fuzzy games is generated when the notion of convexity is considered. Convex fuzzy games can be successfully used for solving sharing problems arising from many economic situations where "cooperation" is the main benefit/cost savings generator; all the solution concepts treated in Chapter 6 have nice properties for such games. Moreover, for convex fuzzy games one can use additional sharing rules which are based on more specific solution concepts like participation monotonic allocation schemes and egalitarian solutions.

7.1 Basic characterizations

Let $v : [0,1]^n \to \mathbb{R}$. Then v satisfies

(i) *supermodularity* (SM) if

$$v\,(s \vee t) + v\,(s \wedge t) \geq v(s) + v(t) \text{ for all } s,t \in [0,1]^N . \tag{7.1}$$

(ii) *coordinate-wise convexity* (CwC) if for each $i \in N$ and each $s^{-i} \in [0,1]^{N \setminus \{i\}}$ the function $g_{s^{-i}} : [0,1] \to \mathbb{R}$ with $g_{s^{-i}}(t) = v(s^{-i} \parallel t)$ for each $t \in [0,1]$ is a convex function (see page 5).

Now we introduce our definition of convex cooperative fuzzy games (cf. [10]).

Definition 7.1. *Let $v \in FG^N$. Then v is called a* ***convex fuzzy game*** *if $v : [0,1]^N \to \mathbb{R}$ satisfies SM and CwC.*

Remark 7.2. Convex fuzzy games form a convex cone.

Remark 7.3. For a weaker definition of a convex fuzzy game we refer to [72], where only the supermodularity property is used.

As shown in Proposition 7.4, a nice example of convex fuzzy games is a unanimity game in which the minimal wining coalition corresponds to a crisp-like coalition.

Proposition 7.4. *Let $u_t \in FG^N$ be the unanimity game based on the fuzzy coalition $t \in \mathcal{F}_0^N$. Then the game u_t is convex if and only if $t = e^T$ for some $T \in 2^N \setminus \{\emptyset\}$.*

Proof. Suppose $t \neq e^T$ for some $T \in 2^N \setminus \{\emptyset\}$. Then there is a $k \in N$ such that $\varepsilon = \min\{t_k, 1 - t_k\} > 0$ and $0 = u_t(t + \varepsilon e^k) - u_t(t) < u_t(t) - u_t(t - \varepsilon e^k) = 1$, implying that u_t is not convex.

Conversely, suppose that $t = e^T$ for some $T \in 2^N \setminus \{\emptyset\}$. Then we show that u_t has the supermodularity property and the coordinate-wise convexity property. Take $s, k \in \mathcal{F}^N$. We can distinguish three cases.

(1) If $u_t(s \vee k) + u_t(s \wedge k) = 2$, then $u_t(s \wedge k) = 1$. Thus, the supermodularity condition (7.1) follows from $u_t(s) + u_t(k) \geq 2u_t(s \wedge k) = 2$.

(2) If $u_t(s \vee k) + u_t(s \wedge k) = 0$, then $u_t(s \vee k) = 0$. Hence $u_t(s) + u_t(k) \leq 2u_t(s \vee k) = 0$, $u_t(s) + u_t(k) = 0$.

(3) If $u_t(s \vee k) + u_t(s \wedge k) = 1$, then $u_t(s \vee k) = 1$ and $u_t(s \wedge k) = 0$. Therefore $u_t(s)$ or $u_t(k)$ must be equal to 0, and, consequently, $u_t(s) + u_t(k) \leq 1$ and the supermodularity condition (7.1) is fulfilled.

Hence, the supermodularity property holds for u_{e^T}.

Secondly, to prove the coordinate-wise convexity of u_{e^T}, note that all functions $g_{s^{-i}}$ in the definition of coordinate-wise convexity are convex because they are either constant with value 0 or with value 1, or they have value 0 on $[0, 1)$ and value 1 in 1. So u_{e^T} is a convex game.

In the following the set of convex fuzzy games with player set N will be denoted by CFG^N. Clearly, $CFG^N \subset FG^N$. Remember that the set of convex crisp games with player set N was denoted by CG^N.

Proposition 7.5. *Let $v \in CFG^N$. Then $cr(v) \in CG^N$.*

Proof. We will prove that $cr(v)$ satisfies SM for crisp games (cf. (4.1)). Take $S, T \in 2^N$ and apply SM for fuzzy games (7.1) with $e^S, e^T, e^{S \cup T}, e^{S \cap T}$ in the roles of $s, t, s \vee t, s \wedge t$, respectively, obtaining

$$cr(v)\,(S \cup T) + cr(v)(S \cap T) \geq cr(v)(S) + cr(v)(T).$$

The next property for convex fuzzy games is related with the increasing marginal contribution property for players in crisp games (cf. Theorem 4.9(iii)). It states that a level increase of a player in a fuzzy coalition has more beneficial effect in a larger coalition than in a smaller coalition.

Proposition 7.6. *Let $v \in CFG^N$. Let $i \in N$, $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \le s^2$ and let $\varepsilon \in \mathbb{R}_+$ with $0 \le \varepsilon \le 1 - s_i^2$. Then*

$$v\left(s^1 + \varepsilon e^i\right) - v\left(s^1\right) \le v\left(s^2 + \varepsilon e^i\right) - v\left(s^2\right). \tag{7.2}$$

Proof. Suppose $N = \{1, \ldots, n\}$. Define the fuzzy coalitions $c^0, c^1, \ldots, c^n$ by $c^0 = s^1$, and $c^k = c^{k-1} + \left(s_k^2 - s_k^1\right) e^k$ for $k \in \{1, \ldots, n\}$. Then $c^n = s^2$. To prove (7.2) it is sufficient to show that for each $k \in \{1, \ldots, n\}$ the inequality $\left(I^k\right)$ holds

$$v\left(c^k + \varepsilon e^i\right) - v\left(c^k\right) \ge v\left(c^{k-1} + \varepsilon e^i\right) - v\left(c^{k-1}\right). \tag{I^k}$$

Note that $\left(I^i\right)$ follows from the coordinate-wise convexity of v and $\left(I^k\right)$ for $k \ne i$ follows from SM with $c^{k-1} + \varepsilon e^i$ in the role of s and c^k in the role of t. Then $s \vee t = c^k + \varepsilon e^i$, $s \wedge t = c^{k-1}$.

Also an analogue of the increasing marginal contribution property for coalitions (cf. Theorem 4.9(ii)) holds as we see in

Proposition 7.7. *Let $v \in CFG^N$. Let $s, t \in \mathcal{F}^N$ and $z \in \mathbb{R}_+^n$ such that $s \le t \le t + z \le e^N$. Then*

$$v(s+z) - v(s) \le v(t+z) - v(t). \tag{7.3}$$

Proof. For each $k \in \{1, \ldots, n\}$ it follows from Proposition 7.6 (with $s + \sum_{r=1}^{k-1} z_r e^r$ in the role of s^1, $t + \sum_{r=1}^{k-1} z_r e^r$ in the role of s^2, k in the role of i, and z_k in the role of ε) that

$$v\left(s + \sum_{r=1}^{k} z_r e^r\right) - v\left(s + \sum_{r=1}^{k-1} z_r e^r\right) \le$$
$$v\left(t + \sum_{r=1}^{k} z_r e^r\right) - v\left(t + \sum_{r=1}^{k-1} z_r e^r\right).$$

By adding these n inequalities we obtain inequality (7.3).

The next proposition introduces a characterizing property for convex fuzzy games which we call the *increasing average marginal return property* (IAMR). This property expresses the fact that for a convex game an increase in participation level of any player in a smaller coalition yields per unit of level less than an increase in a larger coalition under the condition that the reached level of participation in the first case is still not bigger than the reached participation level in the second case.

Proposition 7.8. *Let $v \in CFG^N$. Let $i \in N$, $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \le s^2$ and let $\varepsilon_1, \varepsilon_2 > 0$ with $s_i^1 + \varepsilon_1 \le s_i^2 + \varepsilon_2 \le 1$. Then*

$$\varepsilon_1^{-1}\left(v\left(s^1 + \varepsilon_1 e^i\right) - v\left(s^1\right)\right) \le \varepsilon_2^{-1}\left(v\left(s^2 + \varepsilon_2 e^i\right) - v\left(s^2\right)\right). \tag{7.4}$$

Proof. From Proposition 7.6 (with s^1, $\left(s^2 + \left(s_i^1 - s_i^2\right) e^i\right)$ and ε_1 in the roles of s^1, s^2 and ε, respectively) it follows that

$$\varepsilon_1^{-1} \left(v \left(s^2 + \left(s_i^1 - s_i^2 + \varepsilon_1\right) e^i\right) - v \left(s^2 + \left(s_i^1 - s_i^2\right) e^i\right)\right) \geq \varepsilon_1^{-1} \left(v \left(s^1 + \varepsilon_1 e^i\right) - v \left(s^1\right)\right).$$

Further, from CwC (by noting that $s_i^2 + \varepsilon_2 \geq s_i^2 + \left(s_i^1 - s_i^2 + \varepsilon_1\right)$, $s_i^2 \geq s_i^2 + \left(s_i^1 - s_i^2\right)$) it follows that

$$\varepsilon_2^{-1} \left(v \left(s^2 + \varepsilon_2 e^i\right) - v \left(s^2\right)\right) \geq \varepsilon_1^{-1} \left(v \left(s^2 + \left(s_i^1 - s_i^2 + \varepsilon_1\right) e^i\right) - v \left(s^2 + \left(s_i^1 - s_i^2\right) e^i\right)\right),$$

resulting in the desired inequality.

Theorem 7.9. *Let $v \in FG^N$. Then the following assertions are equivalent:*
(i) $v \in CFG^N$;
(ii) v satisfies IAMR.

Proof. We know from Proposition 7.8 that a convex game satisfies IAMR. On the other hand it is clear that IAMR implies the CwC. Hence, we only have to prove that IAMR implies SM. So, given $s, t \in \mathcal{F}^N$ we have to prove inequality (7.1).

Let $P = \{i \in N \mid t_i < s_i\}$. If $P = \emptyset$, then (7.1) follows from the fact that $s \vee t = t, s \wedge t = s$. For $P \neq \emptyset$, arrange the elements of P in a sequence $\sigma(1), \ldots, \sigma(p)$, where $p = |P|$, and put $\varepsilon_{\sigma(k)} = s_{\sigma(k)} - t_{\sigma(k)} > 0$ for $k \in \{1, \ldots, p\}$. Note that in this case

$$s = s \wedge t + \sum_{k=1}^{p} \varepsilon_{\sigma(k)} e^{\sigma(k)}, \quad s \vee t = t + \sum_{k=1}^{p} \varepsilon_{\sigma(k)} e^{\sigma(k)}.$$

Hence,

$$v(s) - v(s \wedge t) = \sum_{r=1}^{p} \left(v \left(s \wedge t + \sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)} \right) - v \left(s \wedge t + \sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)} \right) \right),$$

and

$$v(s \vee t) - v(t) = \sum_{r=1}^{p} \left(v \left(t + \sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)} \right) - v \left(t + \sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)} \right) \right).$$

From these equalities the relation (7.1) follows because IAMR implies for each $r \in \{1, \ldots, p\}$:

$$v\left(s \wedge t+\sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right)-v\left(s \wedge t+\sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right)$$
$$\leq v\left(t+\sum_{k=1}^{r} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right)-v\left(t+\sum_{k=1}^{r-1} \varepsilon_{\sigma(k)} e^{\sigma(k)}\right).$$

Next, we study the implications of two other properties a function $v : [0,1]^N \to \mathbb{R}$ may satisfy (cf. [67]). The first one we call *monotonicity of the first partial derivatives property* (MOPAD), and the second one is the *directional convexity property* (DICOV) introduced in [41].

Let $i \in N$ and $s \in [0,1]^N$. We say that the left derivative $D_i^- v(s)$ of v in the i-th direction at s with $0 < s_i \leq 1$ exists if $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \varepsilon^{-1}(v(s)-v(s-\varepsilon e^i))$ exists and is finite; then

$$D_i^- v(s) := \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \varepsilon^{-1}(v(s)-v(s-\varepsilon e^i)).$$

Similarly, the right derivative of v in the i-th direction at s with $0 \leq s_i < 1$, denoted by $D_i^+ v(s)$, is $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \varepsilon^{-1}(v(s+\varepsilon e^i)-v(s))$ if this limit exists and is finite. We put for convenience $D_i^- v(s) = -\infty$ if $s_i = 0$ and $D_i^+ v(s) = \infty$ if $s_i = 1$.

Definition 7.10. *We say that $v : [0,1]^N \to \mathbb{R}$ satisfies **MOPAD** if for each $i \in N$ the following four conditions hold:*
(M1) $D_i^- v(s)$ exists for each $s \in [0,1]^N$ with $0 < s_i \leq 1$;
(M2) $D_i^+ v(s)$ exists for each $s \in [0,1]^N$ with $0 \leq s_i < 1$;
(M3) $D_i^- v(s) \leq D_i^+ v(s)$;
(M4) $D_i^- v(s^1) \leq D_i^- v(s^2)$ and $D_i^+ v(s^1) \leq D_i^+ v(s^2)$ for each $s^1, s^2 \in [0,1]^N$ with $s^1 \leq s^2$.

Definition 7.11. *Let $[a,b] = \{x \in [0,1]^N \mid a_i \leq x_i \leq b_i$ for each $i \in N\}$. We say that $v : [0,1]^N \to \mathbb{R}$ satisfies **DICOV** if for each $a, b \in [0,1]^N$ with $a \leq b$ and each pair $c, d \in [a,b]$ with $c+d = a+b$ it follows that*

$$v(a)+v(b) \geq v(c)+v(d).$$

Lemma 7.12. *Let $s \in [0,1]^N$ and $i \in N$ with $0 \leq s_i < 1$ and $0 < \varepsilon_1 \leq \varepsilon_2 < \varepsilon_3 \leq 1-s_i$. If $v : [0,1]^N \to \mathbb{R}$ satisfies MOPAD then*

$$\varepsilon_1^{-1}(v(s+\varepsilon_1 e^i)-v(s)) \leq (\varepsilon_3-\varepsilon_2)^{-1}(v(s+\varepsilon_3 e^i)-v(s+\varepsilon_2 e^i)).$$

Proof. From the definition of the left derivative and by (M4) it follows

$$
\begin{aligned}
\varepsilon_1^{-1}(v(s+\varepsilon_1 e^i)-v(s)) &= \varepsilon_1^{-1}\int_0^{\varepsilon_1} D_i^- v(s+xe^i)dx \\
&\le \varepsilon_1^{-1}\int_0^{\varepsilon_1} D_i^- v(s+\varepsilon_1 e^i)dx = D_i^- v(s+\varepsilon_1 e^i) \\
&\le D_i^- v(s+\varepsilon_2 e^i) = (\varepsilon_3-\varepsilon_2)^{-1}\int_{\varepsilon_2}^{\varepsilon_3} D_i^- v(s+\varepsilon_2 e^i)dx \\
&\le (\varepsilon_3-\varepsilon_2)^{-1}\int_{\varepsilon_2}^{\varepsilon_3} D_i^- v(s+xe^i)dx \\
&= (\varepsilon_3-\varepsilon_2)^{-1}(v(s+\varepsilon_3 e^i)-v(s+\varepsilon_2 e^i)).
\end{aligned}
$$

The following theorem establishes the equivalence among the introduced properties provided that the characteristic function v is continuous.

Theorem 7.13. *Let $v : [0,1]^N \to \mathbb{R}$ be a continuous function. The following assertions are equivalent:*
(i) v satisfies SM and CwC;
(ii) v satisfies MOPAD;
(iii) v satisfies IAMR;
(iv) v satisfies DICOV.

Proof. (i) $\to$ (ii): The validity of (M1), (M2), and (M3) in the definition of MOPAD follows by CwC. To prove (M4) note first that for s^1 with $s_i^1 = 0$ we have $D_i^- v(s^1) = -\infty \le D_i^- v(s^2)$. If $s_i^1 > 0$, then CwC implies (cf. Proposition 7.6) that

$$v(s^1) - v(s^1 - \varepsilon e_i) \le v(s^2) - v(s^2 - \varepsilon e_i)$$

for $\varepsilon > 0$ such that $s_i^1 - \varepsilon \ge 0$. By multiplying the left and right sides of the above inequality with ε^{-1} and then taking the limit for ε going to 0, we obtain $D_i^- v(s^1) \le D_i^- v(s^2)$. The second inequality in (M4) can be proved in a similar way.

(ii) $\to$ (iii): Suppose that v satisfies MOPAD. We have to prove that for each $a, b \in [0,1]^N$ with $a \le b$, each $i \in N$, $\delta \in (0, 1-a_i]$, $\varepsilon \in (0, 1-b_i]$ such that $a_i + \delta \le b_i + \varepsilon \le 1$ it follows that

$$\delta^{-1}(v(a+\delta e^i) - v(a)) \le \varepsilon^{-1}(v(b+\varepsilon e^i) - v(b)). \tag{7.5}$$

Take $a, b, i, \delta, \varepsilon$ as above. Let $c = a + (b_i - a_i)e^i$ and $d = b + (a_i + \delta - b_i)e^i$. We consider two cases:

(a) $a_i + \delta \le b_i$. Then

$$
\begin{aligned}
\delta^{-1}(v(a+\delta e^i)-v(a)) &= \delta^{-1}\int_0^{\delta} D_i^- v(a+xe^i)dx \\
&\le \delta^{-1}\int_0^{\delta} D_i^- v(a+\delta e^i)dx = D_i^- v(a+\delta e^i)
\end{aligned}
$$

$$\begin{aligned}&\le D_i^- v(b)\\&\le \varepsilon^{-1}\int_0^{\varepsilon} D_i^- v(b+xe^i)dx = \varepsilon^{-1}(v(b+\varepsilon e^i)-v(b)),\end{aligned}$$

where the inequalities follow by (M4).

(b) $a_i+\delta \in (b_i, b_i+\varepsilon]$. Then

$$\begin{aligned}\delta^{-1}(v(a+\delta e^i)-v(a)) &= \delta^{-1}(v(c)-v(a)) + \delta^{-1}(v(a+\delta e^i)-v(c))\\&\le \delta^{-1}(c_i-a_i)(a_i+\delta-c_i)^{-1}(v(a+\delta e^i)-v(c)) + \delta^{-1}(v(a+\delta e^i)-v(c))\\&= (a_i+\delta-c_i)^{-1}(v(a+\delta e^i)-v(c)),\end{aligned}$$

where the inequality follows by Lemma 7.12, with $s=a$, $\varepsilon_1=\varepsilon_2=c_i-a_i<\delta=\varepsilon_3$.

Thus we have

$$\delta^{-1}(v(a+\delta e^i)-v(a)) \le (a_i+\delta-c_i)^{-1}(v(a+\delta e^i)-v(c)). \tag{7.6}$$

Similarly, by applying Lemma 7.12 with $s=b$, $\varepsilon_1=\varepsilon_2=d_i-b_i<\varepsilon=\varepsilon_3$, it follows that

$$(d_i-b_i)^{-1}(v(d)-v(b)) \le \varepsilon^{-1}(v(b+\varepsilon e^i)-v(b)). \tag{7.7}$$

Further, using (M4) with $(a_{-i},t)=(a_1,\ldots,a_{i-1},t,a_{i+1},\ldots,a_n)$ and $(b_{-i},t)=(b_1,\ldots,b_{i-1},t,b_{i+1},\ldots,b_n)$ in the role of s^1 and s^2, respectively, and the equalities $a_i+\delta=d_i$ and $c_i=b_i$ we obtain

$$\begin{aligned}v\left(a+\delta e^i\right)-v(c) &= \int_{c_i}^{a_i+\delta} D_i^- v(a_{-i},t)dt\\&\le \int_{b_i}^{d_i} D_i^- v(b_{-i},t)dt \le v(d)-v(b).\end{aligned}$$

Now, since $a_i+\delta-c_i=d_i-b_i$, we have

$$(a_i+\delta-c_i)^{-1}\left(v\left(a+\delta e^i\right)-v(c)\right) \le (d_i-b_i)^{-1}\left(v(d)-v(b)\right). \tag{7.8}$$

Combining (7.6) and (7.7) via (7.8) (by using the transitivity property of the inequality relation), (7.5) follows.

(iii) $\rightarrow$ (iv): Assume that v satisfies IAMR. Take $a,b\in[0,1]^N$ with $a\le b$ and $c,d\in[a,b]$ with $c+d=a+b$. Define $h=b-c$. Then $b=c+h$ and $d=a+h$. We have

$$\begin{aligned}v(b)-v(c) &= \sum_{r=1}^{n}(v(c+\sum_{i=1}^{r}h_ie^i)-v(c+\sum_{i=1}^{r-1}h_ie^i))\\&\ge \sum_{r=1}^{n}(v(a+\sum_{i=1}^{r}h_ie^i)-v(a+\sum_{i=1}^{r-1}h_ie^i)) = f(d)-f(a),\end{aligned}$$

where the inequality follows by applying IAMR n times.

(iv) $\to$ (i): Let v satisfy DICOV. To prove that v satisfies CwC as well, note that for $s^{-i} \in [0,1]^{N\setminus\{i\}}$ and $0 \leq p < \frac{1}{2}(p+q) < q \leq 1$, we have $(s^{-i} \parallel \frac{1}{2}(p+q)) \in [(s^{-i} \parallel p), (s^{-i} \parallel q)]$. So, (iv) with $a = (s^{-i} \parallel p)$, $b = (s^{-i} \parallel q)$, $c = d = (s^{-i} \parallel \frac{1}{2}(p+q))$ implies $v(s^{-i} \parallel p) + v(s^{-i} \parallel q) \geq 2v(s^{-i} \parallel \frac{1}{2}(p+q))$.

To prove that v satisfies SM, let $c, d \in [0,1]^N$. Then $c, d \in [c \wedge d,\, c \vee d]$. By (iv) one obtains $v(c \vee d) + v(c \wedge d) \geq v(c) + v(d)$.

Finally, we introduce a fifth property that allows for a very simple characterization of a convex fuzzy game. It requires that all second partial derivatives of $v : [0,1]^N \to \mathbb{R}$ are non-negative (NNSPAD).

Definition 7.14. *Let $v \in C^2$. Then v satisfies **NNSPAD** on $[0,1]^N$ if for all $i, j \in N$ we have*

$$\frac{\partial^2 v}{\partial s_i \partial s_j} \geq 0.$$

Obviously, the properties MOPAD and NNSPAD are equivalent on the class of C^2-functions.

Remark 7.15. In [57] it is shown that if $v \in C^2$, then DICOV implies NNSPAD.

Remark 7.16. For each $r \in \{1, \ldots, m\}$, let μ_r be defined for each $s \in [0,1]^N$ by $\mu_r(s) = \sum_{i \in N} s_i \mu_r(i)$ with $\mu_r(i) \geq 0$ for each $i \in N$ and $\sum_{i \in N} \mu_r(i) \leq 1$. If $f \in C^1$ satisfies DICOV, then v with $v(s) = f(\mu_1(s), \ldots, \mu_m(s))$ is a convex game (cf. [41]).

7.2 Properties of solution concepts

This section is devoted to special properties of the solution concepts introduced so far on the class of convex fuzzy games. First, we focus on the Aubin core, the fuzzy Shapley value and the fuzzy Weber set. As we see in the following theorem the stable marginal vector property (cf. Theorem 4.9(iv)) also holds for convex fuzzy games and the fuzzy Weber set coincides with the Aubin core. Hence, the Aubin core is large; moreover it coincides with the core of the corresponding crisp game (cf. [10]).

Theorem 7.17. *Let $v \in CFG^N$. Then*
(i) $m^\sigma(v) \in C(v)$ for each $\sigma \in \pi(N)$;
(ii) $C(v) = W(v)$;
(iii) $C(v) = C(cr(v))$.

Proof. (i) For each $\sigma \in \pi(N)$ we have $\sum_{i \in N} m_i^\sigma(v) = v\left(e^N\right)$. Further for each $\sigma \in \pi(N)$ and $s \in \mathcal{F}^N$

$$\begin{aligned}
\sum_{i\in N} s_i m_i^{\sigma}(v) &= \sum_{k=1}^{n} s_{\sigma(k)} m_{\sigma(k)}^{\sigma}(v) \\
&= \sum_{k=1}^{n} s_{\sigma(k)} \left(v\left(\sum_{r=1}^{k} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right)\right) \\
&\geq \sum_{k=1}^{n} \left(v\left(\sum_{r=1}^{k} s_{\sigma(r)} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} s_{\sigma(r)} e^{\sigma(r)}\right)\right) \\
&= v\left(\sum_{r=1}^{n} s_{\sigma(r)} e^{\sigma(r)}\right) = v(s),
\end{aligned}$$

where the inequality follows by applying n times Proposition 7.8. Hence $m^{\sigma}(v) \in C(v)$ for each $\sigma \in \pi(N)$.

(ii) From assertion (i) and the convexity of the core we obtain $W(v) = co\{m^{\sigma}(v) \mid \sigma \in \pi(N)\} \subset C(v)$. The reverse inclusion follows from Proposition 6.19.

(iii) Since $cr(v)$ is a convex crisp game by Proposition 7.5, we have $C(cr(v)) = W(cr(v))$, and $W(cr(v)) = W(v) = C(v)$ by (ii).

It follows from Theorem 7.17 that $\phi(v)$ has a central position in the Aubin core $C(v)$ if v is a convex fuzzy game. For crisp games it holds that a game v is convex if and only if $C(v) = W(v)$ (cf. Theorem 4.9(v)). For fuzzy games the implication is only in one direction. Example 7.18 presents a fuzzy game which is not convex and where the Aubin core and the fuzzy Weber set coincide.

Example 7.18. Let $v \in FG^{\{1,2\}}$ with $v(s_1, s_2) = s_1 s_2$ if $(s_1, s_2) \neq \left(\frac{1}{2}, \frac{1}{2}\right)$ and $v\left(\frac{1}{2}, \frac{1}{2}\right) = 0$. Then $v \notin CFG^{\{1,2\}}$, but $C(v) = W(v) = co\{(0,1), (1,0)\}$.

Example 7.19. Consider the public good game in Example 5.5. If the functions $g_1, \ldots, g_n$ and $-k$ are convex, then we have $v \in CFG^N$.

For fuzzy games the core is a superadditive solution, i.e.

$$C(v+w) \supset C(v) + C(w) \text{ for all } v, w \in FG^N,$$

and the fuzzy games with a non-empty Aubin core form a cone.

On the set of convex fuzzy games the Aubin core turns out to be an additive correspondence as we see in

Proposition 7.20. *The Aubin core of a convex fuzzy game and the fuzzy Shapley value are additive solutions.*

Proof. Let v, w be convex fuzzy games. Then

$$\begin{aligned}
C(v+w) &= C(cr(v+w)) = C(cr(v) + cr(w)) \\
&= C(cr(v)) + C(cr(w)) = C(v) + C(w),
\end{aligned}$$

where the first equality follows from Theorem 7.17(iii) and the third equality follows from the additivity of the core for convex crisp games (cf. [13]). Further from $\phi(v) = \phi(cr(v))$ and the additivity of the Shapley value for convex crisp games it follows that $\phi(v+w) = \phi(v) + \phi(w)$.

Now we study properties of other cores and stable sets for convex fuzzy games (cf. [70]).

Lemma 7.21. *Let $v \in CFG^N$. Take $x, y \in I(v)$ and suppose $x \operatorname{dom}_s y$ for some $s \in \mathcal{F}_0^N$. Then $|car(s)| \geq 2$.*

Proof. Take $x, y \in I(v)$ and suppose $x \operatorname{dom}_s y$ for some $s \in \mathcal{F}_0^N$ with $car(s) = \{i\}$. Then $x_i > y_i$ and $s_i x_i \leq v(s_i e^i)$. By the convexity of v, we obtain $s_i v(e^i) \geq v(s_i e^i)$. Thus we have $y_i < x_i \leq \frac{v(s_i e^i)}{s_i} \leq v(e^i)$ which is a contradiction with the individual rationality of y.

Theorem 7.22. *Let $v \in CFG^N$ and $w = cr(v)$. Then, for all $x, y \in I(v) = I(w)$, we have $x \operatorname{dom} y$ in v if and only if $x \operatorname{dom} y$ in w.*

Proof. First we note that
$$I(v) = \left\{x \in \mathbb{R}^n \mid \textstyle\sum_{i \in N} x_i = v\left(e^N\right),\ x_i \geq v(e^i) \text{ for each } i \in N\right\}$$
and
$$I(w) = \left\{x \in \mathbb{R}^n \mid \textstyle\sum_{i \in N} x_i = w\left(N\right),\ x_i \geq w(\{i\}) \text{ for each } i \in N\right\}$$
coincide because $w(N) = v(e^N)$ and $w(i) = v(e^i)$ for each $i \in N$.

To prove the 'if' part, let $x, y \in I(w) = I(v)$ and $x \operatorname{dom}_S y$ for some $S \in 2^N \setminus \{\emptyset\}$. Then it implies $x \operatorname{dom}_{e^S} y$ in v.

Now we prove the 'only if' part. Let $x, y \in I(v) = I(w)$ and $x \operatorname{dom}_s y$ for some $s \in \mathcal{F}_0^N$. Let $\varphi(s) = |\{i \in N \mid 0 < s_i < 1\}|$. By Remark 6.5, $\varphi(s) < n$. It is sufficient to prove by induction on $\varphi(s) \in \{0, \ldots, n-1\}$, that $x \operatorname{dom}_s y$ implies $x \operatorname{dom} y$ in w.

Clearly, if $\varphi(s) = 0$ then $x \operatorname{dom}_{car(s)} y$ because $\varphi(s) = 0$ implies that s is a crisp-like coalition.

Suppose now that the assertion "$x \operatorname{dom}_s y$ in v with $\varphi(s) = k$ implies $x \operatorname{dom} y$ in w" holds for each k with $0 \leq k < r < n$. Take $s \in \mathcal{F}_0^N$ with $\varphi(s) = r$, and $i \in N$ such that $0 < s_i < 1$, and take $x, y \in I(v)$ such that $x \operatorname{dom}_s y$. Then $x_i > y_i$ for each $i \in car(s)$ and $s \cdot x \leq v(s)$. Further, $|car(s)| \geq 2$ by Lemma 7.21. We note that s can be represented by a convex combination of $a = s - s_i e^i$ and $b = s + (1 - s_i)e^i$, i.e. $s = (1 - s_i)a + s_i b$. Note that $\varphi(a) = r - 1$ and $\varphi(b) = r - 1$. Further, $|car(a)| = |car(s)| - 1$, $|car(b)| = |car(s)|$.

The inequality $s \cdot x \leq v(s)$ implies $(1 - s_i)a \cdot x + s_i b \cdot x \leq v(s)$. On the other hand, the (coordinate-wise) convexity of v induces $v(s) \leq (1 - s_i)v(a) + s_i v(b)$.

Hence, $(1 - s_i)a \cdot x + s_i b \cdot x \leq v(s) \leq (1 - s_i)v(a) + s_i v(b)$ which implies $(1 - s_i)(a \cdot x - v(a)) + s_i(b \cdot x - v(b)) \leq 0$; thus $a \cdot x \leq v(a)$ or $b \cdot x \leq v(b)$. We want to show that

$$x \operatorname{dom}_a y \text{ or } x \operatorname{dom}_b y. \tag{7.9}$$

The following three cases should be considered:

(1) $b \cdot x \leq v(b)$. Then $x \operatorname{dom}_b y$, since $|car(b)| \geq 2$.

(2) $b \cdot x > v(b)$ and $|car(s)| \geq 3$. Then $a \cdot x \leq v(a)$; thus $x \operatorname{dom}_a y$, since $|car(a)| \geq 2$.

(3) $b \cdot x > v(b)$ and $|car(s)| = 2$. Then we have $a \cdot x \leq v(a)$ and $|car(b)| = 1$.

By the convexity of v and the individual rationality, we obtain $a \cdot x \geq v(a)$. In fact, let $a = s_j e^j$. Then the convexity of v induces $s_j v(e^j) \geq v(s_j e^j)$. By the individual rationality, we obtain $x_j \geq v(e^j)$. Hence, $a \cdot x = s_j x_j \geq s_j v(e^j) \geq v(s_j e^j) = v(a)$. So, $a \cdot x = v(a)$, which is contradictory to $(1 - s_i)(a \cdot x - v(a)) + s_i(b \cdot x - v(b)) \leq 0$, implying that case (3) does not occur.

Hence, (7.9) holds. Since $\varphi(a) = \varphi(b) = r - 1$, the induction hypothesis implies that $x \operatorname{dom} y$ in w.

Theorem 7.23. *Let $v \in CFG^N$. Then*
(i) $C(v) = C^P(v) = C^{cr}(v)$;
(ii) $DC(v) = DC(cr(v))$;
(iii) $C(v) = DC(v)$.

Proof. (i) For convex fuzzy games $C(v) = C(cr(v))$ (see Theorem 7.17(iii)). Now, we use Theorem 6.8(i).

(ii) From Theorem 7.22 we conclude that $DC(v) = DC(cr(v))$.

(iii) Since $v \in CFG^N$ we have $v(e^N) \geq v(s) + \sum_{i \in N \setminus car(s)} v(e^i)$ for each $s \in \mathcal{F}^N$. We obtain by Theorem 6.9 that $C^P(v) = DC(v)$. Now we use (i).

The next theorem extends the result in [60] that each crisp convex game has a unique stable set coinciding with the dominance core.

Theorem 7.24. *Let $v \in CFG^N$. Then there is a unique stable set, namely $DC(v)$.*

Proof. By Theorem 2.11(iii), $DC(cr(v))$ is the unique stable set of $cr(v)$. In view of Theorem 7.22, the set of stable sets of v and $cr(v)$ coincide, and by Theorem 7.23(ii), we have $DC(v) = DC(cr(v))$. So, the unique stable set of v is $DC(v)$.

Note that the game v in Example 6.14 is convex, but v_1 and v_2 are not.

Remark 7.25. The relations among the Aubin core, the proper core, the crisp core, the dominance core, and the stable sets discussed above remain valid if one uses the corresponding generalized versions of these notions as introduced and studied in [36].

We describe now the implications of convexity on the structure of the tight catchers and compromise values introduced in Sections 6.4 and 6.5.

Let $D_k v(0)$ and $D_k v\left(e^N\right)$ for each $k \in N$ be the right and left partial derivative in the direction e^k in 0 and e^N, respectively.

Theorem 7.26. *Let $v \in CFG^N$. Then*

$$HQ(v) = \left[Dv(0), Dv\left(e^N\right)\right].$$

Proof. From the fact that v satisfies IAMR (cf. Proposition 7.8) it follows that

$$l_k\left(Q(v)\right) = inf\left\{\varepsilon^{-1}\left(v\left(\varepsilon e^k\right) - v(0)\right) \mid \varepsilon \in (0,1]\right\} = D_k v(0),$$

and

$$u_k\left(Q(v)\right) = \sup\left\{\varepsilon^{-1}\left(v\left(e^N\right) - v\left(e^N - \varepsilon e^k\right)\right) \mid \varepsilon \in (0,1]\right\} = D_k v\left(e^N\right).$$

Theorem 7.27. *Let $v \in CFG^N$. Then $HC(v) = HW(v)$ and this hypercube is a tight catcher for $C(v) = W(v)$. Further*

$$l_k\left(C(v)\right) = v\left(e^k\right),$$
$$u_k\left(C(v)\right) = v\left(e^N\right) - v\left(e^{N\setminus\{k\}}\right)$$

for each $k \in N$.

Proof. For $v \in CFG^N$ the IAMR property implies

$$\varepsilon^{-1}\left(v\left(\varepsilon e^k\right) - v(0)\right) \leq v\left(e^k\right) - v(0) \text{ for each } \varepsilon \in (0,1]$$

and

$$v\left(e^k\right) - v(0) \leq v\left(e^S + e^k\right) - v\left(e^S\right) \text{ for each } S \subset N \setminus \{k\}.$$

The first inequality corresponds to $s^1 = s^2 = 0$, $\varepsilon_1 = \varepsilon$, and $\varepsilon_2 = 1$, while the second inequality is obtained by taking $s^1 = 0$, $s^2 = e^S$, and $\varepsilon_1 = \varepsilon_2 = 1$.

So, we obtain

$$\begin{aligned} l_k\left(C(v)\right) &= \sup\left\{\varepsilon^{-1} v\left(\varepsilon e^k\right) \mid \varepsilon \in (0,1]\right\} = v\left(e^k\right) \\ &= \min\left\{v\left(e^{S\cup\{k\}}\right) - v\left(e^S\right) \mid S \subset N \setminus \{k\}\right\} \\ &= l_k\left(W(v)\right). \end{aligned}$$

Similarly, from IAMR it follows

$$\begin{aligned} u_k\left(C(v)\right) &= \inf\left\{\varepsilon^{-1}\left(v\left(e^N\right) - v\left(e^N - \varepsilon e^k\right)\right) \mid \varepsilon \in (0,1]\right\} \\ &= v\left(e^N\right) - v\left(e^{N\setminus\{k\}}\right) \\ &= \max\left\{v\left(e^{S\cup\{k\}}\right) - v\left(e^S\right) \mid S \subset N \setminus \{k\}\right\} \\ &= u_k\left(W(v)\right). \end{aligned}$$

This implies that $HC(v) = HW(v)$.

That this hypercube is a tight catcher of $C(v) = W(v)$ (see Theorem 7.17(ii)) follows from the facts that

$$l_k(W(v)) = v\left(e^k\right) = m_k^\sigma(v),$$
$$u_k(W(v)) = v\left(e^N\right) - v\left(e^{N\setminus\{k\}}\right) = m_k^\tau(v),$$

where σ and τ are orderings of N with $\sigma(1) = k$ and $\tau(n) = k$, respectively.

For convex fuzzy games this theorem has consequences with respect to the coincidence of some of the compromise values introduced in Section 6.5. This can be seen in our next theorem.

Theorem 7.28. *Let $v \in CFG^N$. Then*
(i) $m_k^v(u(C(v))) = m_k^v(u(W(v))) = v\left(e^k\right)$ for each $k \in N$;
(ii) $val_C^\tau(v) = val_C^\sigma(v) = val_W^\tau(v) = val_W^\sigma(v)$.

Proof. (i) By Theorem 7.27, $u_k(C(v)) = u_k(W(v)) = v\left(e^N\right) - v\left(e^{N\setminus\{k\}}\right)$ for each $k \in N$. So, to prove (i), we have to show that for $k \in N$,

$$\begin{aligned} m_k^v(u(C(v))) &= \sup\left\{s_k^{-1}\left(v(s) - \int_{j\in N\setminus\{k\}} s_j\left(v\left(e^N\right) - v\left(e^{N\setminus\{j\}}\right)\right)\right)\right\} \\ &= v\left(e^k\right), \end{aligned}$$

where the sup is taken over $s \in \mathcal{F}^N$, $s_k > 0$.

Equivalently, we have to show that for each $s \in \mathcal{F}^N$ with $s_k > 0$

$$s_k v\left(e^k\right) \geq \int_{j\in N\setminus\{k\}} s_j\left(v\left(e^N\right) - v\left(e^{N\setminus\{j\}}\right)\right). \tag{7.10}$$

Let σ be an ordering on N with $\sigma(1) = k$. Then

$$\begin{aligned} v(s) &= \int_{t=1}^{n}\left(v\left(\int_{r=1}^{t} s_{\sigma(r)}e^{\sigma(r)}\right) - v\left(\int_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right) \\ &= v\left(s_k e^k\right) + \int_{t=2}^{n}\left(v\left(\int_{r=1}^{t} s_{\sigma(r)}e^{\sigma(r)}\right) - v\left(\int_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right). \end{aligned}$$

Now, note that for each $t \in \{2, \ldots, n\}$ IAMR implies

$$\begin{aligned} &s_{\sigma(t)}^{-1}\left(v\left(\int_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)} + s_{\sigma(t)}e^{\sigma(t)}\right) - v\left(\int_{r=1}^{t-1} s_{\sigma(r)}e^{\sigma(r)}\right)\right) \\ &\quad \leq v\left(e^{N\setminus\{\sigma(t)\}} + 1.e^{\sigma(t)}\right) - v\left(e^{N\setminus\{\sigma(t)\}}\right). \end{aligned}$$

So, we obtain

$$v(s) \leq s_k v\left(e^k\right) + \int_{j\in N\setminus\{k\}} s_j \left(v\left(e^N\right) - v\left(e^{N\setminus\{j\}}\right)\right)$$

from which (7.10) follows.

(ii) Since, by (i) and Theorem 7.27, $l_k\left(C(v)\right) = m_k^v\left(u\left(C(v)\right)\right) = v\left(e^k\right)$ for each $k \in N$, it follows that $val_C^\tau(v) = val_C^\sigma(v) = val_W^\tau(v) = val_W^\sigma(v)$.

Remark 7.29. Let $v \in CFG^N$. Because $u\left(Q(v)\right) \geq u\left(C(v)\right)$, it follows from (i) in the proof of Theorem 7.28 that $m_k\left(u\left(Q(v)\right)\right) = v\left(e^k\right)$ for each $k \in N$. But in general this remainder vector is not equal to $Dv(0)$ (cf. Theorem 7.26), so in general $val_Q^\tau(v)$ and $val_Q^\sigma(v)$ do not coincide.

Example 7.30. Let $v \in CFG^{\{1,2\}}$ with $v\left(s_1, s_2\right) = s_1\left(s_2\right)^5$ for each $s = (s_1, s_2) \in \mathcal{F}^{\{1,2\}}$. Then, by Theorems 7.17(ii) and 7.27, $C(v) = W(v) = conv\left\{m^{(1,2)}(v), m^{(2,1)}(v)\right\} = \{(0,1),(1,0)\}$ and $HC(v) = HW(v) = [(0,0),(1,1)]$. Hence, $val_C^\sigma(v) = val_W^\sigma(v) = \left(\frac{1}{2},\frac{1}{2}\right)$. Further, $val_Q^\sigma(v) = \left(\frac{1}{6},\frac{5}{6}\right)$ because, by Theorem 7.26, $HQ(v) = \left[Dv(e^\emptyset), Dv\left(e^{\{1,2\}}\right)\right] = [(0,0), (1,5)]$. By Theorem 7.28, $val_C^\tau(v) = val_W^\tau(v) = \left(\frac{1}{2},\frac{1}{2}\right) = val_C^\sigma(v) = val_W^\sigma(v)$. Further, $val_Q^\tau(v)$ is the compromise between $m^v(1,5) = (0,0)$ and $(1,5)$, so in this case also $val_Q^\tau(v) = val_Q^\sigma(v) = \left(\frac{1}{6},\frac{5}{6}\right)$.

7.3 Participation monotonic allocation schemes

In this section we introduce for convex fuzzy games the notion of a *participation monotonic allocation scheme* (*pamas*). This notion is inspired by [63] where population monotonic allocation schemes (pmas) for cooperative crisp games which are necessarily totally balanced (cf. Subsections 4.1.2 and 4.2.2) are introduced. Recall that a pmas for a crisp game is a bundle of core elements, one for each subgame and the game itself, which are related via a monotonicity condition guaranteeing that each player is better off when more other players join him. In our approach (cf. [10]) the role of subgames of a crisp game will be taken over by the t-restricted games $v_t \in FG^N$ of a fuzzy game $v \in FG^N$ (cf. Definition 5.6).

Remark 7.31. Note that for each core element $x \in C(v_t)$ we have $x_i = 0$ for each $i \notin car(t)$. This follows from

$$\begin{aligned}0 &= v(e^\emptyset) = v_t(e^i)\\ &\leq x_i = \sum_{k\in N} x_k - \sum_{k\in N\setminus\{i\}} x_k \leq v_t(e^N) - v_t(e^{N\setminus\{i\}}) = 0,\end{aligned}$$

where we use that $i \notin car(t)$ in the second and last equalities, and that $x \in C(v_t)$ in the two inequalities.

Remark 7.32. If $v \in CFG^N$, then also $v_t \in CFG^N$ for each $t \in \mathcal{F}^N$.

Definition 7.33. *Let $v \in FG^N$. A scheme $(a_{i,t})_{i \in N, t \in \mathcal{F}_0^N}$ is called a* ***participation monotonic allocation scheme*** *(pamas) if*
(i) $(a_{t,i})_{i \in N} \in C(v_t)$ for each $t \in \mathcal{F}_0^N$ (stability condition);
(ii) $t_i^{-1} a_{t,i} \geq s_i^{-1} a_{s,i}$ for each $s, t \in \mathcal{F}_0^N$ with $s \leq t$ and each $i \in car(s)$ (participation monotonicity condition).

Remark 7.34. Note that such a pamas is an $n \times \infty$-matrix, where the columns correspond to the players and the rows to the fuzzy coalitions. In each row corresponding to t there is a core element of the game v_t. The participation monotonicity condition implies that, if the scheme is used as regulator for the payoff distributions in the restricted fuzzy games, players are paid per unit of participation more in larger coalitions than in smaller coalitions.

Remark 7.35. Note that the collection of participation monotonic allocation schemes of a fuzzy game v is a convex set of $n \times \infty$-matrices.

Remark 7.36. In [72] inspired by [63], the notion of fuzzy population monotonic allocation scheme (FPMAS) is introduced. The relation between such a scheme and core elements is not studied there.

Remark 7.37. A necessary condition for the existence of a pamas for v is the existence of core elements for v_t for each $t \in \mathcal{F}_0^N$. But this is not sufficient as Example 7.38 shows. A sufficient condition is the convexity of a game as we see in Theorem 7.40.

Example 7.38. Consider the game $v \in FG^N$ with $N = \{1,2,3,4\}$ and $v(s) = \min\{s_1 + s_2, s_3 + s_4\}$ for each $s = (s_1, s_2, s_3, s_4) \in \mathcal{F}^N$. Suppose for a moment that $(a_{i,t})_{i \in N, t \in \mathcal{F}_0^N}$ is a pamas. Then for $t^1 = e^{N \setminus \{2\}}$, $t^2 = e^{N \setminus \{1\}}$, $t^3 = e^{N \setminus \{4\}}$, and $t^4 = e^{N \setminus \{3\}}$ we have $C(v_{t^k}) = \{e^k\}$ (see Example 5.3), and so $(a_{i,t^k})_{i \in N} = e^k$ for $k \in N$. But then $\sum_{k \in N} a_{k,e^N} \geq \sum_{k \in N} a_{k,t^k} = 4 > 2 = v(e^N)$, and this implies that there does not exist a pamas. Note that $C(v_t) \neq \emptyset$ holds for any $t = (t_1, t_2, t_3, t_4) \in \mathcal{F}_0^N$, because $(t_1, t_2, 0, 0) \in C(v_t)$ if $t_1 + t_2 \leq t_3 + t_4$; and $(0, 0, t_3, t_4) \in C(v_t)$ otherwise.

Definition 7.39. *Let $v \in FG^N$ and $x \in C(v)$. Then we call x* ***pamas extendable*** *if there exists a pamas $(a_{i,t})_{i \in N, t \in \mathcal{F}_0^N}$ such that $a_{i,e^N} = x_i$ for each $i \in N$.*

In the next theorem we see that convex games have a pamas. Moreover, each core element is pamas extendable.

Theorem 7.40. *Let $v \in CFG^N$ and $x \in C(v)$. Then x is pamas-extendable.*

Proof. We know from Theorem 7.17 that x is in the convex hull of the marginal vectors $m^{\sigma}(v)$ with $\sigma \in \pi(N)$. In view of Remark 7.35 we only need to prove that each marginal vector $m^{\sigma}(v)$ is pamas extendable, because then the right convex combination of these pamas extensions gives a pamas extension of x.

So take $\sigma \in \pi(N)$ and define $(a_{i,t})_{i \in N, t \in \mathcal{F}_0^N}$ by $a_{i,t} = m_i^{\sigma}(v_t)$ for each $i \in N$, $t \in \mathcal{F}_0^N$. We claim that this scheme is a pamas extension of $m^{\sigma}(v)$.

Clearly, $a_{i,e^N} = m_i^{\sigma}(v)$ for each $i \in N$ since $v_{e^N} = v$. Further, by Remark 7.32, each t-restricted game v_t is a convex fuzzy game, and from Theorem 7.17 it follows that $(a_{i,t})_{i \in N} \in C(v_t)$. Hence the scheme satisfies the stability condition.

To prove the participation monotonicity condition, take $s, t \in \mathcal{F}_0^N$ with $s \leq t$ and $i \in car(s)$ and let k be the element in N such that $i = \sigma(k)$. We have to prove that $t_i^{-1} a_{i,t} \geq s_i^{-1} a_{i,s}$. Now

$$\begin{aligned}
t_i^{-1} a_{i,t} &= t_{\sigma(k)}^{-1} m_{\sigma(k)}^{\sigma}(v_t) \\
&= t_{\sigma(k)}^{-1} \left(v \left(\sum_{r=1}^{k} t_{\sigma(r)} e^{\sigma(r)} \right) - v \left(\sum_{r=1}^{k-1} t_{\sigma(r)} e^{\sigma(r)} \right) \right) \\
&\geq s_{\sigma(k)}^{-1} \left(v \left(\sum_{r=1}^{k} s_{\sigma(r)} e^{\sigma(r)} \right) - v \left(\sum_{r=1}^{k-1} s_{\sigma(r)} e^{\sigma(r)} \right) \right) \\
&= s_{\sigma(k)}^{-1} m_{\sigma(k)}^{\sigma}(v_s) = s_i^{-1} a_{i,s},
\end{aligned}$$

where the inequality follows from the convexity of v (i.e. v satisfies IAMR). So $(a_{i,t})_{i \in N, t \in \mathcal{F}_0^N}$ is a pamas extension of $m^{\sigma}(v)$.

Further, the *total fuzzy Shapley value* of a game $v \in CFG^N$, which is the scheme $(\phi_{i,t})_{i \in N, t \in \mathcal{F}_0^N}$ with the fuzzy Shapley value of the restricted game v_t in each row corresponding to t, is a pamas. The total fuzzy Shapley value is a Shapley function (in the sense of [72]) on the class of n-person fuzzy games. For a study of a Shapley function in relation with FPMAS we refer the reader to [72].

Example 7.41. Let $v \in FG^{\{1,2\}}$ be given by $v(s_1, s_2) = 4s_1(s_1 - 2) + 10(s_2)^2$ for each $s = (s_1, s_2) \in \mathcal{F}^{\{1,2\}}$. Then v is convex and $m^{(1,2)}(v) = m^{(2,1)}(v) = \phi(v) = (-4, 10)$ because in fact v is additive: $v(s_1, s_2) = v(s_1, 0) + v(0, s_2)$. For each $t \in \mathcal{F}_0^N$ the fuzzy Shapley value $\phi(v_t)$ equals $(4t_1(t_1 - 2), 10(t_2)^2)$, and the scheme $(a_{i,t})_{i \in \{1,2\}, t \in \mathcal{F}_0^N}$ with $a_{1,t} = 4t_1(t_1 - 2)$, $a_{2,t} = 10(t_2)^2$ is a pamas extension of $\phi(v)$, with the fuzzy Shapley value of v_t in each row corresponding to t of the scheme, so $(a_{i,t})_{i \in \{1,2\}, t \in \mathcal{F}_0^N}$ is the total fuzzy Shapley value of v.

7.4 Egalitarianism in convex fuzzy games

In this section we are interested in introducing an egalitarian solution for convex fuzzy games. We do this in a constructive way by adjusting the classical Dutta-Ray algorithm for a convex crisp game (cf. [26]).

As mentioned in Subsection 4.2.3, at each step of the Dutta-Ray algorithm for convex crisp games a largest element exists. Note that for the crisp case the supermodularity of the characteristic function is equivalent to the convexity of the corresponding game.

Although the cores of a convex fuzzy game and its related (convex) crisp game coincide and the Dutta-Ray constrained egalitarian solution is a core element, finding the egalitarian solution of a convex fuzzy game is a task on itself. As we show in Lemma 7.42, supermodularity of a fuzzy game implies a semilattice structure of the corresponding (possibly infinite) set of fuzzy coalitions with maximal average worth (cf. (5.1)), but it is not enough to ensure the existence of a maximal element. Different difficulties which can arise in fuzzy games satisfying only the supermodularity property are illustrated by means of three examples. According to Lemma 7.46 it turns out that adding coordinate-wise convexity to supermodularity is sufficient for the existence of such a maximal element; moreover, this element corresponds to a crisp coalition. Then, a simple method becomes available to calculate the egalitarian solution of a convex fuzzy game (cf. [11]).

Lemma 7.42. *Let $v \in FG^N$ be a supermodular game. Then the set*

$$A(N,v) := \left\{ t \in \mathcal{F}_0^N \mid \alpha(t,v) = \sup_{s \in \mathcal{F}_0^N} \alpha(s,v) \right\}$$

is closed with respect to the join operation $\vee$.

Proof. Let $\overline{\alpha} = \sup_{s \in \mathcal{F}_0^N} \alpha(s,v)$. If $\overline{\alpha} = \infty$, then $A(N,v) = \emptyset$, so $A(N,v)$ is closed w.r.t. the join operation.

Suppose now $\overline{\alpha} \in \mathbb{R}$. Take $t^1, t^2 \in A(N,v)$. We have to prove that $t^1 \vee t^2 \in A(N,v)$, that is $\alpha\left(t^1 \vee t^2, v\right) = \overline{\alpha}$.

Since $v\left(t^1\right) = \overline{\alpha}\left\lceil t^1 \right\rfloor$ and $v\left(t^2\right) = \overline{\alpha}\left\lceil t^2 \right\rfloor$ we obtain

$$\begin{aligned}
\overline{\alpha}\left\lceil t^1 \right\rfloor + \overline{\alpha}\left\lceil t^2 \right\rfloor &= v\left(t^1\right) + v\left(t^2\right) \\
&\leq v\left(t^1 \vee t^2\right) + v\left(t^1 \wedge t^2\right) \\
&\leq \overline{\alpha}\left\lceil t^1 \vee t^2 \right\rfloor + \overline{\alpha}\left\lceil t^1 \wedge t^2 \right\rfloor = \overline{\alpha}\left\lceil t^1 \right\rfloor + \overline{\alpha}\left\lceil t^2 \right\rfloor,
\end{aligned}$$

where the first inequality follows from SM and the second inequality follows from the definition of $\overline{\alpha}$ and the fact that $v(e^{\emptyset}) = 0$. This implies that $v\left(t^1 \vee t^2\right) = \overline{\alpha}\left\lceil t^1 \vee t^2 \right\rfloor$, so $t^1 \vee t^2 \in A(N,v)$.

We can conclude from the proof of Lemma 7.42 that in case $t^1, t^2 \in A(N,v)$ not only $t^1 \vee t^2 \in A\,(N,v)$ but also $t^1 \wedge t^2 \in A\,(N,v)$ if $t^1 \wedge t^2 \neq e^{\emptyset}$. Further, $A\,(N,v)$ is closed w.r.t. finite "unions", where $t^1 \vee t^2$ is seen as the "union" of t^1 and t^2.

If we try to introduce in a way similar to that of [26] an egalitarian rule for supermodular fuzzy games, then problems may arise since the set of non-empty fuzzy coalitions is infinite and it is not clear if there exists a maximal fuzzy coalition with "maximum value per unit of participation level". To be more precise, if $v \in FG^N$ is a supermodular fuzzy game then crucial questions are:

(1) Is $\sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ finite or not? Example 7.43 presents a fuzzy game for which $\sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ is infinite.

(2) In case that $\sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ is finite, is there a $t \in \mathcal{F}_0^N$ s.t. $\alpha\,(t,v) = \sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$? A fuzzy game for which the set $\arg\sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ is empty is given in Example 7.44. Note that if the set $\arg\sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ is non-empty then $\sup_{s \in \mathcal{F}_0^N} \alpha\,(s,v) = \max_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$.

(3) Let $\geq$ be the standard partial order on $[0,1]^N$. Suppose that $\max_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ exists. Does the set $\arg\max_{s \in \mathcal{F}_0^N} \alpha\,(s,v)$ have a maximal element in $\mathcal{F}_0^N$ w.r.t. $\geq$? That this does not always hold for a fuzzy game is shown in Example 7.45.

Example 7.43. Let $v \in FG^{\{1,2\}}$ be given by

$$v(s_1,s_2) = \begin{cases} s_2 tg \frac{\pi s_1}{2} & \text{if } s_1 \in [0,1)\,, \\ 0 & \text{otherwise,} \end{cases}$$

for each $s = (s_1,s_2) \in \mathcal{F}^{\{1,2\}}$. For this game $\sup_{s \in \mathcal{F}_0^{\{1,2\}}} \alpha\,(s,v) = \infty$.

Example 7.44. Let $v \in FG^{\{1,2,3\}}$ with

$$v(s_1,s_2,s_3) = \begin{cases} (s_1+s_2+s_3)^2 & \text{if } s_1,s_2,s_3 \in [0,1)\,, \\ 0 & \text{otherwise,} \end{cases}$$

for each $s = (s_1,s_2,s_3) \in \mathcal{F}^{\{1,2,3\}}$. For this game $\sup_{s \in \mathcal{F}_0^{\{1,2,3\}}} \alpha\,(s,v) = 3$, and $\arg\sup_{s \in \mathcal{F}_0^{\{1,2,3\}}} \alpha\,(s,v) = \emptyset$.

Example 7.45. Let $v \in FG^{\{1,2\}}$ be given by

$$v(s_1,s_2) = \begin{cases} s_1+s_2 & \text{if } s_1,s_2 \in [0,1)\,, \\ 0 & \text{otherwise,} \end{cases}$$

for each $s = (s_1,s_2) \in \mathcal{F}^{\{1,2\}}$. For this game $\max_{s \in \mathcal{F}_0^{\{1,2\}}} \alpha\,(s,v) = 1$, $\arg\max_{s \in \mathcal{F}_0^{\{1,2\}}} \alpha\,(s,v) = [0,1) \times [0,1) \setminus \{0\}$, but this set has no maximal element w.r.t. $\geq$.

One can easily check that the games in Examples 7.43, 7.44, and 7.45 are supermodular, but not convex (CwC is not satisfied). For convex fuzzy games all three questions mentioned above are answered affirmatively in Theorem 7.48. By using this theorem, the following additional problems can also be overcome: "How to define the reduced games in the steps of the adjusted algorithm, and whether this algorithm has only a finite number of steps?"

In the proof of Lemma 7.46 we will use the notion of degree of fuzziness of a coalition (cf. page 50). Note that for $s \in \mathcal{F}_0^N$ with degree of fuzziness $\varphi(s) = 0$ we have $\alpha(s, v) \leq \max_{S \in 2^N \setminus \{\emptyset\}} \alpha\left(e^S, v\right)$, because s is equal to e^T for some $T \in 2^N \setminus \{\emptyset\}$.

Lemma 7.46. *Let $v \in CFG^N$ and $s \in \mathcal{F}_0^N$. If $\varphi(s) > 0$, then there is a $t \in \mathcal{F}_0^N$ with $\varphi(t) = \varphi(s) - 1$, $car(t) \subset car(s)$, and $\alpha(t, v) \geq \alpha(s, v)$; if $\alpha(t, v) = \alpha(s, v)$ then $t \geq s$.*

Proof. Take $s \in \mathcal{F}_0^N$ with $\varphi(s) > 0$, and $i \in N$ such that $s_i \in (0, 1)$. Consider $t^0 = \left(s^{-i}, 0\right)$ and $t^1 = \left(s^{-i}, 1\right)$. Note that $\varphi\left(t^0\right) = \varphi\left(t^1\right) = \varphi(s) - 1$ and $car(t^0) \subset car(t^1) = car(s)$.

If $t^0 = e^{\emptyset}$, then $t^1 = e^i$ and then $\alpha\left(e^i, v\right) \geq \alpha\left(s_i e^i, v\right) = \alpha(s, v)$ follows from CwC. We then take $t = e^i$.

If $t^0 \neq e^{\emptyset}$ and $\alpha\left(t^0, v\right) > \alpha(s, v)$, then we take $t = t^0$.

Now we treat the case $t^0 \neq e^{\emptyset}$ and $\alpha\left(t^0, v\right) \leq \alpha(s, v)$. From the last inequality and from the fact that $\frac{v(s)}{\lceil s \rfloor}$ is a convex combination of $\frac{v(t^0)}{\lceil t^0 \rfloor}$ and $\frac{v(s)-v(t^0)}{\lceil s-t^0 \rfloor}$, i.e.

$$\alpha(s, v) = \frac{v(s)}{\lceil s \rfloor} = \frac{\lceil t^0 \rfloor}{\lceil s \rfloor} . \frac{v\left(t^0\right)}{\lceil t^0 \rfloor} + \frac{\lceil s - t^0 \rfloor}{\lceil s \rfloor} . \frac{v(s) - v\left(t^0\right)}{\lceil s - t^0 \rfloor},$$

we obtain

$$\frac{v(s) - v\left(t^0\right)}{\lceil s - t^0 \rfloor} \geq \frac{v(s)}{\lceil s \rfloor} = \alpha(s, v). \tag{7.11}$$

From the fact that v satisfies IAMR (with $t^0, s, \lceil s - t^0 \rfloor, \lceil t^1 - s \rfloor$ in the roles of $s^1, s^2, \varepsilon_1, \varepsilon_2$, respectively) it follows

$$\frac{v\left(t^1\right) - v(s)}{\lceil t^1 - s \rfloor} \geq \frac{v(s) - v\left(t^0\right)}{\lceil s - t^0 \rfloor}. \tag{7.12}$$

Now from (7.11) and (7.12) we have

$$\frac{v\left(t^1\right) - v(s)}{\lceil t^1 - s \rfloor} \geq \frac{v(s)}{\lceil s \rfloor} = \alpha(s, v). \tag{7.13}$$

Then, by applying (7.13), we obtain

$$\alpha\left(t^1, v\right) = \frac{v\left(t^1\right)}{\lceil t^1 \rfloor} = \frac{\lceil t^1 - s \rfloor}{\lceil t^1 \rfloor} \cdot \frac{v\left(t^1\right) - v\left(s\right)}{\lceil t^1 - s \rfloor} + \frac{\lceil s \rfloor}{\lceil t^1 \rfloor} \cdot \frac{v\left(s\right)}{\lceil s \rfloor}$$
$$\geq \frac{\lceil t^1 - s \rfloor}{\lceil t^1 \rfloor} \cdot \frac{v\left(s\right)}{\lceil s \rfloor} + \frac{\lceil s \rfloor}{\lceil t^1 \rfloor} \cdot \frac{v\left(s\right)}{\lceil s \rfloor} = \frac{v\left(s\right)}{\lceil s \rfloor} = \alpha\left(s, v\right).$$

So, we can take $t = t^1$.

From Lemma 7.46 it follows that for each $s \in \mathcal{F}_0^N$, there is a sequence $s^0, \ldots, s^k$ in $\mathcal{F}_0^N$, where $s^0 = s$ and $k = \varphi(s)$ such that $\varphi\left(s^{r+1}\right) = \varphi\left(s^r\right) - 1$, $car\left(s^{r+1}\right) \subset car\left(s^r\right)$, and $\alpha\left(s^{r+1}, v\right) \geq \alpha\left(s^r, v\right)$ for each $r \in \{0, \ldots, k-1\}$. Since $\varphi\left(s^k\right) = 0$, s^k corresponds to a crisp coalition, say T. So, we have proved

Corollary 7.47. *Let $v \in CFG^N$. Then for all $s \in \mathcal{F}_0^N$ there exists $T \in 2^N \setminus \{\emptyset\}$ such that $T \subset car(s)$ and $\alpha\left(e^T, v\right) \geq \alpha\left(s, v\right)$.*

From Corollary 7.47 it follows immediately

Theorem 7.48. *Let $v \in CFG^N$. Then*
(i) $\sup_{s \in \mathcal{F}_0^N} \alpha\left(s, v\right) = \max_{T \in 2^N \setminus \{\emptyset\}} \alpha\left(e^T, v\right)$;
(ii) $T^* = \max\left(\arg\max_{T \in 2^N \setminus \{\emptyset\}} \alpha\left(e^T, v\right)\right)$ *generates the largest element in* $\arg\sup_{s \in \mathcal{F}_0^N} \alpha\left(s, v\right)$, *namely* e^{T^*}.

In view of this result it is easy to adjust the Dutta-Ray algorithm to a convex fuzzy game v. In Step 1 one puts $N_1 := N$, $v_1 := v$ and considers $\arg\sup_{s \in \mathcal{F}_0^{N_1}} \alpha\left(s, v_1\right)$. According to Theorem 7.48, there is a unique maximal element in $\arg\sup_{s \in \mathcal{F}_0^{N_1}} \alpha\left(s, v\right)$, which corresponds to a crisp coalition, say S_1. Define $E_i\left(v\right) = \alpha\left(e^{S_1}, v_1\right)$ for each $i \in S_1$. If $S_1 = N$, then we stop.

In case $S_1 \neq N$, then in Step 2 one considers the convex fuzzy game v_2 with $N_2 := N_1 \setminus S_1$ and, for each $s \in [0,1]^{N \setminus S_1}$,

$$v_2\left(s\right) = v_1\left(e^{S_1} \curvearrowright s\right) - v_1\left(e^{S_1}\right),$$

where $\left(e^{S_1} \curvearrowright s\right)$ is the element in $[0,1]^N$ with

$$\left(e^{S_1} \curvearrowright s\right)_i = \begin{cases} 1 & \text{if } i \in S_1, \\ s_i & \text{if } i \in N \setminus S_1. \end{cases}$$

Once again, by using Theorem 7.48, one can take the largest element e^{S_2} in $\arg\max_{S \in 2^{N_2} \setminus \{\emptyset\}} \alpha\left(e^S, v_2\right)$ and defines $E_i\left(v\right) = \alpha\left(e^{S_2}, v_2\right)$ for all $i \in S_2$. If $S_1 \cup S_2 = N$ we stop; otherwise we continue by considering the convex fuzzy game v_3, etc. After a finite number of steps the algorithm stops, and the obtained allocation $E\left(v\right)$ is called *the egalitarian solution of the convex fuzzy game v.*

Theorem 7.49. *Let $v \in CFG^N$. Then*
(i) $E(v) = E(cr(v))$;
(ii) $E(v) \in C(v)$;
(iii) $E(v)$ Lorenz dominates every other allocation in the Aubin core $C(v)$.

Proof. (i) This assertion follows directly from Theorem 7.48 and the adjusted Dutta-Ray algorithm given above.

(ii) Note that $E(v) = E(cr(v)) \in C(cr(v)) = C(v)$, where the first equality follows from (i), the second equality follows from Theorem 7.17(iii), and the relation $E(cr(v)) \in C(cr(v))$ is a main result in [26] for convex crisp games.

(iii) It is a fact that $E(cr(v))$ Lorenz dominates every other element of $C(cr(v))$ (cf. [26]). Since $E(v) = E(cr(v))$ and $C(cr(v)) = C(v)$, our assertion (iii) follows.

Theorem 7.49 should be interpreted as strengthening Dutta and Ray's result. One can also think that the egalitarian solution for a convex crisp game will keep the Lorenz domination property in any fuzzy extension satisfying IAMR.

The Dutta-Ray egalitarian solution for convex fuzzy games is also related to the equal division core (cf. page 59) for convex fuzzy games as Theorem 7.50 shows.

Theorem 7.50. *Let $v \in CFG^N$. Then*
(i) $C(v) \subset EDC(v)$;
(ii) $E(v) \in EDC(v)$;
(iii) $EDC(v) = EDC(cr(v))$.

Proof. (i) Suppose $x \notin EDC(v)$. Then there exists $s \in \mathcal{F}_0^N$ s.t. $\alpha(s, v) > x_i$ for all $i \in car(s)$. Then

$$\sum_{i=1}^{n} s_i x_i < \sum_{i=1}^{n} \alpha(s, v) s_i = v(s)$$

which implies that $x \notin C(v)$. So $C(v) \subseteq EDC(v)$.

(ii) According to (i) and Theorem 7.49(ii), we have $E(v) \in C(v) \subseteq EDC(v)$.

(iii) The relation $EDC(v) \subset EDC(cr(v))$ follows from Proposition 6.16. Suppose $x \in EDC(cr(v))$. We prove that for each $s \in \mathcal{F}_0^N$ there is $i \in car(s)$ s.t. $x_i \geq \alpha(s, v)$.

Take T as in Corollary 7.47. Since $x \in EDC(cr(v))$, there is an $i \in T$ s.t. $x_i \geq \alpha\left(e^T, v\right)$. Now, from Corollary 7.47 it follows that $x_i \geq \alpha(s, v)$ for $i \in T \subset car(s)$.

The next example is meant to illustrate the various interrelations among the egalitarian solution, the core, and the equal division core for convex fuzzy games as stated in Theorems 7.49 and 7.50.

Example 7.51. Let $N = \{1,2,3\}$ and $T = \{1,2\}$. Consider the unanimity fuzzy game u_{e^T} with

$$u_{e^T}(s) = \begin{cases} 1 \text{ if } s_1 = s_2 = 1, \\ 0 \text{ otherwise,} \end{cases}$$

for each $s = (s_1, s_2, s_3) \in \mathcal{F}^{\{1,2,3\}}$. According to Proposition 7.4, the game u_{e^T} is convex. Its Aubin core is given by

$$C(u_{e^T}) = co\left\{e^1, e^2\right\} = co\left\{(1,0,0), (0,1,0)\right\},$$

and the egalitarian allocation is given by

$$E(u_{e^T}) = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \in C(u_{e^T}).$$

It is easy to see that $E(u_{e^T})$ Lorenz dominates every other allocation in $C(u_{e^T})$. Moreover, the equal division core $EDC(u_{e^T})$ is the set $B_1 \cup B_2$, where $B_1 = co\left\{e^1, \frac{1}{2}\left(e^1 + e^2\right), \frac{1}{2}\left(e^1 + e^3\right)\right\}$ and $B_2 = co\left\{\frac{1}{2}\left(e^1 + e^2\right), e^2, \frac{1}{2}\left(e^2 + e^3\right)\right\}$. Note that $C(u_{e^T}) \subset EDC(u_{e^T}) = EDC(cr(u_{e^T}))$.

8
Fuzzy clan games

In this chapter we consider fuzzy games of the form $v : [0,1]^{N_1} \times \{0,1\}^{N_2} \to \mathbb{R}$, where the players in N_1 have participation levels which may vary between 0 and 1, while the players in N_2 are crisp players in the sense that they can fully cooperate or not cooperate at all. With this kind of games we can model various economic situations where the group of agents involved is divided into two subgroups with different status: a "clan" consisting of "powerful" agents and a set of available agents willing to cooperate with the clan. This cooperation generates a positive reward only for coalitions where all clan members are present. Such situations are modeled in the classical theory of cooperative games with transferable utility by means of (total) clan games where only the full cooperation and non-cooperation at all of non-clan members with the clan are taken into account (cf. Section 4.3). Here we take over this simplifying assumption and allow non-clan members to cooperate with all clan members and some other non-clan members to a certain extent. As a result the notion of a fuzzy clan game is introduced.

8.1 The cone of fuzzy clan games

Let $N = \{1, ..., n\}$ be a finite set of players. We denote the non-empty set of clan members by C, and treat clan members as crisp players. In the following we denote the set of crisp subcoalitions of C by $\{0,1\}^C$, the set of fuzzy coalitions on $N \backslash C$ by $[0,1]^{N \backslash C}$ (equivalent to $\mathcal{F}^{N \backslash C}$), and denote $[0,1]^{N \backslash C} \times \{0,1\}^C$ by $\mathcal{F}^N_C$. For each $s \in \mathcal{F}^N_C$, $s_{N \backslash C}$ and s_C will denote its restriction to $N\backslash C$ and C, respectively. We denote the vector $(e^N)_C$ by 1_C in the following. Further we denote by $\mathcal{F}^N_{1_C}$ the set $[0,1]^{N\backslash C} \times \{1_C\}$ of fuzzy coalitions on N where all clan members have participation level 1, and where the participation level of non-clan members may vary between 0 and 1 (cf. [71]).

We define fuzzy clan games by using veto power of clan members, monotonicity, and a condition reflecting the fact that a decrease in participation level of a non-clan member in growing coalitions containing at least all clan members with full participation level results in a decrease of the average marginal return of that player (DAMR-property).

Formally, a game $v : \mathcal{F}^N_C \to \mathbb{R}$ is a *fuzzy clan game* if v satisfies the following three properties:

(i) (*veto-power of clan members*) $v(s) = 0$ if $s_C \neq 1_C$;

(ii) (*Monotonicity*) $v(s) \leq v(t)$ for all $s, t \in \mathcal{F}^N_C$ with $s \leq t$;

(iii) (*DAMR-property for non-clan members*) For each $i \in N\backslash C$, all $s^1, s^2 \in \mathcal{F}^N_{1_C}$ and all $\varepsilon_1, \varepsilon_2 > 0$ such that $s^1 \leq s^2$ and $0 \leq s^1 - \varepsilon_1 e^i \leq s^2 - \varepsilon_2 e^i$ we have

$$\varepsilon_1^{-1}(v(s^1) - v(s^1 - \varepsilon_1 e^i)) \geq \varepsilon_2^{-1}(v(s^2) - v(s^2 - \varepsilon_2 e^i)).$$

Property (i) expresses the fact that the full participation level of all clan members is a necessary condition for generating a positive reward for coalitions.

Fuzzy clan games for which the clan consists of a single player are called *fuzzy big boss games*, with the single clan member as the big boss.

Remark 8.1. One can see a fuzzy clan game as a special mixed action-set game, the latter being introduced in [19].

As an introduction we give two examples of interactive situations one of them leading to a fuzzy clan game, but the other one not.

Example 8.2. (A production situation with owners and gradually available workers) Let $N\backslash C = \{1, \ldots, m\}, C = \{m+1, \ldots, n\}$. Let $f : [0,1]^{N\backslash C} \to \mathbb{R}$ be a monotonic non-decreasing function with $f(0) = 0$ that satisfies the DAMR-property. Then $v : [0,1]^{N\backslash C} \times \{0,1\}^C \to \mathbb{R}$ defined by $v(s) = 0$ if $s_C \neq 1_C$ and $v(s) = f(s_1, \ldots, s_m)$ otherwise, is a fuzzy clan game with clan C. One can think of a production situation where the clan members are providers of different (complementary) essential tools needed for the production and the production function measures the gains if all clan members are cooperating with the set of workers $N\backslash C$ (cf. [52]), where each worker i can participate at level s_i which may vary from lack of participation to full participation.

Example 8.3. (A fuzzy voting situation with a fixed group with veto power) Let N and C be as in Example 8.2, and $0 < k < |N\backslash C|$. Let $v : [0,1]^{N\backslash C} \times \{0,1\}^C \to \mathbb{R}$ with

$$v(s) = \begin{cases} 1 \text{ if } s_C = 1_C \text{ and } \sum_{i=1}^m s_i \geq k, \\ 0 \text{ otherwise.} \end{cases}$$

Then v has the veto power property for members in C and the monotonicity property, but not the DAMR-property with respect to members of $N \backslash C$, hence it is not a fuzzy clan game. This game can be seen as arising from a voting situation where to pass a bill all members of C have to (fully) agree upon and the sum of the support levels $\sum_{i \in N \setminus C} s_i$ of $N \setminus C$ should exceed a fixed threshold k, where $s_i = 1$ ($s_i = 0$) correspond to full support (no support) of the bill, but also partial supports count.

In the following the set of all fuzzy clan games with a fixed non-empty set of players N and a fixed clan C is denoted by FCG_C^N. We notice that FCG_C^N is a convex cone in FG^N, that is for all $v, w \in FCG_C^N$ and $p, q \in \mathbb{R}_+$, $pv + qw \in FCG_C^N$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers.

Now we show that for each game $v \in FCG_C^N$ the corresponding crisp game $w = cr(v)$ is a total clan game if $|C| \geq 2$, and a total big boss game if $|C| = 1$.

Let $v \in FCG_C^N$. The corresponding crisp game w has the following properties which follow straightforwardly from the properties of v:

- $w(S) = 0$ if $C \not\subset S$;
- $w(S) \leq w(T)$ for all S, T with $S \subset T \subset N$;
- for all S, T with $C \subset S \subset T$ and each $i \in S \setminus C$, $w(S) - w(S \backslash \{i\}) \geq w(T) - w(T \setminus \{i\})$.

So, w is a total clan game in the terminology of [73] if $|\, C \,| \geq 2$ (cf. Subsection 4.3.2) and a total big boss game in the terminology of [14] if $|\, C \,| = 1$.

Fuzzy clan games can be seen as an extension of crisp clan games in what concerns the possibilities of cooperation available to non-clan members. Specifically, in a fuzzy clan game each non-clan member can be involved in cooperation at each extent between 0 and 1, whereas in a crisp clan game a non-clan member can only be or not a member of a (crisp) coalition containing all clan members.

In the following we consider t-restricted games corresponding to a fuzzy clan game and prove, in Proposition 8.4, that these games are also fuzzy clan games.

Let $v \in FCG_C^N$ and $t \in \mathcal{F}_{1_C}^N$. Recall that the t-restricted game v_t of v with respect to t is given by $v_t(s) = v(t * s)$ for each $s \in \mathcal{F}_C^N$.

Proposition 8.4. *Let v_t be the t-restricted game of $v \in FCG_C^N$, with $t \in \mathcal{F}_{1_C}^N$. Then $v_t \in FCG_C^N$.*

Proof. First, note that for each $s \in \mathcal{F}_C^N$ with $s_C \neq 1_C$ we have $(t{*}s)_C \neq 1_C$, and then the veto-power property of v implies $v_t(s) = v(t * s) = 0$. To prove the monotonicity property, let $s^1, s^2 \in \mathcal{F}_C^N$ with $s^1 \leq s^2$. Then

$v_t(s^1) = v(t * s^1) \leq v(t * s^2) = v_t(s^2)$, where the inequality follows from the monotonicity of v. Now, we focus on DAMR regarding non-clan members. Let $i \in N \backslash C$, $s^1, s^2 \in \mathcal{F}^N_{1_C}$, and let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $s^1 \leq s^2$ and $0 \leq s^1 - \varepsilon_1 e^i \leq s^2 - \varepsilon_2 e^i$. Then

$$\begin{aligned} \varepsilon_2^{-1}(v_t(s^2) - v_t(s^2 - \varepsilon_2 e^i)) &= \varepsilon_2^{-1}(v(t * s^2) - v(t * s^2 - t_i \varepsilon_2 e^i)) \\ &\leq \varepsilon_1^{-1}(v(t * s^1) - v(t * s^1 - t_i \varepsilon_1 e^i)) \\ &= \varepsilon_1^{-1}(v_t(s^1) - v_t(s^1 - \varepsilon_1 e^i), \end{aligned}$$

where the inequality follows from the fact that v satisfies the DAMR-property.

For each $i \in N \backslash C$, $x \in [0,1]$ and $t \in \mathcal{F}^N_C$, let $(t^{-i} \,||\, x)$ be the element in $\mathcal{F}^N_C$ such that $(t^{-i} \,||\, x)_j = t_j$ for each $j \in N \backslash \{i\}$ and $(t^{-i} \,||\, x)_i = x$. The function $v : [0,1]^{N \backslash C} \times \{0,1\}^C \to \mathbb{R}$ is called *coordinate-wise concave regarding non-clan members* if for each $i \in N \backslash C$ the function $g_{t^{-i}} : [0,1] \to \mathbb{R}$ with $g_{t^{-i}}(x) = v(t^{-i} \,||\, x)$ for each $x \in [0,1]$ is a concave function. Intuitively, this means that the function v is concave in each coordinate corresponding to (the participation level of) a non-clan member when all other coordinates are kept fixed.

The function $v : [0,1]^{N \backslash C} \times \{0,1\}^C \to \mathbb{R}$ is said to have the *submodularity property on* $[0,1]^{N \backslash C}$ if $v(s \vee t) + v(s \wedge t) \leq v(s) + v(t)$ for all $s, t \in \mathcal{F}^N_{1_C}$, where $s \vee t$ and $s \wedge t$ are those elements of $[0,1]^{N \backslash C} \times \{1_C\}$ with the i-th coordinate equal, for each $i \in N \setminus C$, to $\max\{s_i, t_i\}$ and $\min\{s_i, t_i\}$, respectively.

Remark 8.5. The DAMR-property regarding non-clan members implies two important properties of v, namely coordinate-wise concavity and submodularity. Note that the coordinate-wise concavity follows straightforwardly from the DAMR-property of v. The proof of the submodularity follows the same line as in the proof of Theorem 7.9 where it was shown that the IAMR-property implies supermodularity.

Let $\varepsilon > 0$ and $s \in \mathcal{F}^N_C$. For each $i \in N \setminus C$ we denote by $D_i v(s)$ the i-th left derivative of v in s if $s_i > 0$, and the i-th right derivative of v in s if $s_i = 0$, i.e. $D_i v(s) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v(s) - v(s - \varepsilon\, e^i))$, if $s_i > 0$, and $D_i v(s) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v(s + \varepsilon\, e^i) - v(s))$, if $s_i = 0$. It is well known that for a concave real-valued function each tangent line to the graph lies above the graph of the function. Based on this property we state

Lemma 8.6. *Let $v \in FCG^N_C$, $t \in \mathcal{F}^N_{1_C}$, and $i \in N \backslash C$. Then, for $s_i \in [0, t_i]$, $v(t^{-i} \,||\, t_i) - v(t^{-i} || \, s_i) \geq (t_i - s_i) D_i v(t)$.*

Proof. Applying the coordinate-wise concavity of v and the property of tangent lines to the graph of g_{-i} in $(t_i,\ g_{-i}(t_i))$ one obtains $v(t^{-i} || \, t_i) - (t_i - s_i) D_i v(t) \geq v(t^{-i} || \, s_i)$.

8.2 Cores and stable sets for fuzzy clan games

We provide an explicit description of the Aubin core of a fuzzy clan game and give some insight into its geometrical shape (cf. [70] and [71]). We start with the following

Lemma 8.7. *Let $v \in FCG_C^N$ and $s \in \mathcal{F}_{1_C}^N$. Then $v(e^N) - v(s) \geq \sum_{i \in N \setminus C} (1 - s_i) D_i v(e^N)$.*

Proof. Suppose that $|N \setminus C| = m$ and denote $N \setminus C = \{1, \ldots, m\}$, $C = \{m+1, \ldots, n\}$. Let $a^0, \ldots, a^m$ and $b^1, \ldots, b^m$ be the sequences of fuzzy coalitions on N given by $a^0 = e^N$, $a^r = e^N - \sum_{k=1}^{r} (1 - s_k) e^k$, $b^r = e^N - (1 - s_r) e^r$ for each $r \in \{1, \ldots, m\}$. Note that $a^m = s \in \mathcal{F}_{1_C}^N$, and $a^{r-1} \vee b^r = e^N$, $a^{r-1} \wedge b^r = a^r$ for each $r \in \{1, \ldots, m\}$. Then

$$v(e^N) - v(s) = \sum_{r=1}^{m} \left(v\left(a^{r-1}\right) - v\left(a^r\right) \right) \geq \sum_{r=1}^{m} (v(e^N) - v(b^r)), \qquad (8.1)$$

where the inequality follows from the submodularity property of v applied for each $r \in \{1, \ldots, m\}$. Now, for each $r \in \{1, \ldots, m\}$, we have by Lemma 8.6

$$D_r v(e^N) \leq (1 - s_r)^{-1} (v(e^N) - v(e^N - (1 - s_r) e^r)),$$

thus obtaining

$$v(e^N) - v(b^r) = v(e^N) - v(e^N - (1 - s_r) e^r) \geq (1 - s_r) D_r v(e^N). \qquad (8.2)$$

Now we combine (8.1) and (8.2).

Theorem 8.8. *Let $v \in FCG_C^N$. Then*
(i) $C(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(e^N),\ 0 \leq x_i \leq D_i v(e^N)$ for each $i \in N \setminus C,\ 0 \leq x_i$ for each $i \in C\}$, if $|C| > 1$;
(ii) $C(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(e^N),\ 0 \leq x_i \leq D_i v(e^N)$ for each $i \in N \setminus \{n\},\ v(e^n) \leq x_n\}$, if $C = \{n\}$.

Proof. We only prove (i).

(a) Let $x \in C(v)$. Then $x_i = e^i \cdot x \geq v(e^i) = 0$ for each $i \in N$ and $\sum_{i=1}^{n} x_i = v(e^N)$. Further, for each $i \in N \setminus C$ and each $\varepsilon \in (0, 1)$, we have

$$x_i = \varepsilon^{-1} (e^N \cdot x - (e^N - \varepsilon\, e^i) \cdot x) \leq \varepsilon^{-1} (v(e^N) - v(e^N - \varepsilon\, e^i)).$$

We use now the monotonicity property and the coordinate-wise concavity property of v obtaining that $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-1} (v(e^N) - v(e^N - \varepsilon e^i))$ exists and this limit is equal to $D_i v(e^N)$. Hence $x_i \leq D_i v(e^N)$, thus implying that $C(v)$ is a subset of the set on the right side of the equality in (i).

(b) To prove the converse inclusion, let $x \in \mathbb{R}^n$ with $\sum_{i=1}^n x_i = v(e^N)$, $0 \le x_i \le D_i v(e^N)$ for each $i \in N \setminus C$, and $0 \le x_i$ for each $i \in C$. We have to show that the inequality $s \cdot x \ge v(s)$ holds for each $s \in [0,1]^N$. First, if $s \in [0,1]^N$ is such that $s_C \ne 1_C$, then $v(s) = 0 \le s \cdot x$. Now let $s \in [0,1]^N$, with $s_C = 1_C$. Then

$$\begin{aligned} s \cdot x = \sum_{i \in C} x_i + \sum_{i \in N \setminus C} s_i x_i &= v(e^N) - \sum_{i \in N \setminus C} (1 - s_i) x_i \\ &\ge v(e^N) - \sum_{i \in N \setminus C} (1 - s_i) D_i v(e^N). \end{aligned}$$

The inequality $s \cdot x \ge v(s)$ follows then from Lemma 8.7.

The Aubin core of a fuzzy clan game has an interesting geometric shape. It is the intersection of a simplex with "hyperbands" corresponding to the non-clan members. To be more precise, for fuzzy clan games, we have $C(v) = \Delta(v(e^N)) \cap B_1(v) \cap \cdots \cap B_m(v)$, where $\Delta(v(e^N))$ is the simplex $\{x \in \mathbb{R}^n_+ \mid \sum\limits_{i=1}^n x_i = v(e^N)\}$, and for each player $i \in \{1, \ldots, m\}$, $B_i(v) = \{x \in \mathbb{R}^n \mid 0 \le x_i \le D_i v(e^N)\}$ is the region between the two parallel hyperplanes in $\mathbb{R}^n$, $\{x \in \mathbb{R}^n \mid x_i = 0\}$ and $\{x \in \mathbb{R}^n \mid x_i = D_i v(e^N)\}$, which we call the "hyperband" corresponding to i. An interesting core element is

$$b(v) = \left(\frac{D_1 v(e^N)}{2}, \ldots, \frac{D_m v(e^N)}{2}, t, \ldots, t \right),$$

with

$$t = |C|^{-1} \left(v(e^N) - \sum_{i=1}^m \frac{D_i v(e^N)}{2} \right),$$

which corresponds to the point with a central location in this geometric structure. Note that $b(v)$ is in the intersection of middle-hyperplanes of all hyperbands $B_i(v)$, $i = 1, \ldots, m$, and it has the property that the coordinates corresponding to clan members are equal.

Example 8.9. For a three-person fuzzy big boss game with player 3 as the big boss and $v(e^3) = 0$ the Aubin core has the shape of a parallelogram (in the imputation set) with vertices: $(0, 0, v(e^N))$, $(D_1 v(e^N), 0, v(e^N) - D_1 v(e^N))$, $(0, D_2 v(e^N), v(e^N) - D_2 v(e^N))$, $(D_1 v(e^N), D_2 v(e^N), v(e^N) - D_1 v(e^N) - D_2 v(e^N))$. Note that

$$b(v) = \left(\frac{D_1 v(e^N)}{2}, \frac{D_2 v(e^N)}{2}, v\left(e^N\right) - \frac{D_1 v(e^N) + D_2 v(e^N)}{2} \right)$$

is the middle point of this parallelogram.

For $v \in CFG^N$ we know that $C(v) = C(cr(v))$ (cf. Theorem 7.17(iii)). This is not the case in general for fuzzy clan games as the next example shows.

Example 8.10. Let $N = \{1,2\}$, let $v : [0,1] \times \{0,1\} \to \mathbb{R}$ be given by $v(s_1, 1) = \sqrt{s_1}$, $v(s_1, 0) = 0$ for each $s_1 \in [0,1]$, and let $w = cr(v)$. Then v is a fuzzy big boss game with player 2 as the big boss, and $C(v) = \left\{(\alpha, 1-\alpha) \mid \alpha \in \left[0, \frac{1}{2}\right]\right\}$, $C(w) = \{(\alpha, 1-\alpha) \mid \alpha \in [0,1]\}$. So, $C(v) \neq C(w)$.

The next lemma plays a role in what follows.

Lemma 8.11. *Let $v \in FCG_C^N$. Let $t \in \mathcal{F}_{1_C}^N$ and v_t be the t-restricted game of v. Then, for each non-clan member $i \in car(t)$, $D_i v_t(e^N) = t_i D_i v(t)$.*

Proof. We have that

$$\begin{aligned} D_i v_t(e^N) &= \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v_t(e^N) - v_t(e^N - \varepsilon\, e^i)) \\ &= \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-1}(v(t) - v(t - \varepsilon t_i e^i) \\ &= t_i D_i v(t). \end{aligned}$$

Theorem 8.12. *Let $v \in FCG_C^N$. Then for each $t \in \mathcal{F}_{1_C}^N$ the Aubin core $C(v_t)$ of the t-restricted game v_t is described by*
(i) $C(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(t), 0 \leq x_i \leq t_i D_i v(t)$ for each $i \in N \setminus C$, $0 \leq x_i$ for each $i \in C\}$, if $|C| > 1$;
(ii) $C(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(t), 0 \leq x_i \leq t_i D_i v(t)$ for each $i \in N \setminus \{n\}$, $v(t_n e^n) \leq x_n\}$, if $C = \{n\}$.

Proof. We only prove (i). Let $t \in \mathcal{F}_{1_C}^N$, with $|C| > 1$. Then, by the definition of the Aubin core of a fuzzy game, $C(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v_t(e^N), \sum_{i \in N} s_i x_i \geq v_t(s)$ for each $s \in \mathcal{F}_C^N\}$. Since $v_t(e^N) = v(t)$ and since, by Proposition 8.4, v_t is itself a fuzzy clan game, we can apply Theorem 8.8(i), thus obtaining $C(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(t), 0 \leq x_i \leq D_i v_t(e^N)$ for each $i \in N \setminus C$, $0 \leq x_i$ for each $i \in C\}$. Now we apply Lemma 8.11.

In addition to the interrelations among the different core notions and stable sets for general fuzzy games (cf. Section 6.2) the dominance core and the proper core of a fuzzy clan game coincide.

Theorem 8.13. *Let $v \in FCG_C^N$. Then $DC(v) = C^P(v)$.*

Proof. From the veto-power property we have that $v(e^i) = 0$ for each $i \in N$ if $|C| > 1$. Then the monotonicity of v implies $v(e^N) - \sum_{i \in N \setminus car(s)} v(e^i) - v(s) = v(e^N) - v(s) \geq 0$ for each $s \in \mathcal{F}^N$. One can easily check that in the case $|C| = 1$, $v(e^N) - \sum_{i \in N \setminus car(s)} v(e^i) - v(s) \geq 0$ for each $s \in \mathcal{F}^N$, too. The equality $DC(v) = C^P(v)$ follows then from Theorem 6.8(ii).

Now we give two examples of fuzzy clan games v to illustrate situations in which $DC(v) \neq C(v)$ and $DC(v)$ is not a stable set, respectively.

Example 8.14. Let $N = \{1,2\}$ and let $v : [0,1] \times \{0,1\} \to \mathbb{R}$ be given for all $s_1 \in [0,1]$ by $v(s_1,1) = \sqrt{s_1}$ and $v(s_1,0) = 0$. This is a big boss game with player 2 as the big boss, so $C(v) \neq \emptyset$. Moreover, as in Example 6.15, we obtain $C(v) = \left\{x \in I(v) \mid 0 \leq x_1 \leq \frac{1}{2}\right\}$, $DC(v) = \{(x_1,x_2) \in \mathbb{R}^2_+ \mid x_1 + x_2 = 1\}$, so $DC(v) \neq C(v)$. Note that $I(v) = \{(x_1,x_2) \in \mathbb{R}^2_+ \mid x_1 + x_2 = 1\}$ is the unique stable set.

The following example shows that $DC(v)$ can be a proper subset of a stable set.

Example 8.15. Let $N = \{1,2,3\}$ and v be given by $v(s_1,s_2,0) = 0$ and $v(s_1,s_2,1) = \min\{s_1 + s_2, 1\}$ for all $(s_1,s_2) \in [0,1]^2$. Then $DC(v) = \{(0,0,1)\}$, and no element in $I(v)$ is dominated by $(0,0,1)$. So, $DC(v)$ is not a stable set. The set $K^{a,b} = \{(\varepsilon a, \varepsilon b, 1-\varepsilon) \mid 0 \leq \varepsilon \leq 1\}$ when $a,b \in \mathbb{R}_+$ with $a+b=1$ is a stable set of v.

8.3 Bi-monotonic participation allocation rules

We present in this section the fuzzy counterpart of a bi-monotonic allocation scheme (bi-mas) for total clan games (cf. Section 4.3.2). We call the corresponding scheme a *bi-monotonic participation allocation scheme* (bi-pamas) and study this kind of schemes with the help of a compensation-sharing rule we introduce now (cf. [71]).

Let $N \setminus C = \{1, \cdots, m\}$ and $C = \{m+1, \ldots, n\}$. We introduce for each $\alpha \in [0,1]^m$ and $\beta \in \Delta(C) = \Delta(\{m+1,\ldots,n\}) = \{z \in \mathbb{R}^{n-m}_+, \sum_{i=m+1}^{n} z_i = 1\}$ an allocation rule $\psi^{\alpha,\beta} : FCG^N_C \to \mathbb{R}^n$ whose i-th coordinate $\psi^{\alpha,\beta}_i(v)$ for each $v \in FCG^N_C$ is given by

$$\begin{cases} \alpha_i D_i v\left(e^N\right) & \text{if } i \in \{1,\ldots,m\}, \\ \beta_i \left(v\left(e^N\right) - \sum_{k=1}^m \alpha_k D_k v\left(e^N\right)\right) & \text{if } i \in \{m+1,\ldots,n\}. \end{cases}$$

We call this rule the *compensation-sharing rule with compensation vector α and sharing vector β*. The i-th coordinate α_i of the compensation vector α indicates that player $i \in \{1,\ldots,m\}$ obtains the part $\alpha_i D_i v(e^N)$ of his marginal contribution $D_i v(e^N)$ to e^N. Then for each $i \in \{m+1,\ldots,n\}$, the i-th coordinate β_i of the sharing vector β determines the share

$$\beta_i \left(v\left(e^N\right) - \sum_{k=1}^{m} \alpha_k D_k v\left(e^N\right)\right)$$

for the clan member i from what is left for the group of clan members in e^N.

Theorem 8.16. *Let $v \in FCG_C^N$. Then*
(i) $\psi^{\alpha,\beta}$: $FCG_C^N \to \mathbb{R}^n$ is stable (i.e. $\psi^{\alpha,\beta}(v) \in C(v)$ for each $v \in FCG_C^N$) and additive for each $\alpha \in [0,1]^m$ and each $\beta \in \Delta(C)$;
(ii) $C(v) = \{\psi^{\alpha,\beta}(v) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\}$;
(iii) the multi-function $C : FCG_C^N \to\to \mathbb{R}^n$ which assigns to each $v \in FCG_C^N$ the subset $C(v)$ of $\mathbb{R}^n$ is additive.

Proof. (i) $\psi^{\alpha,\beta}(pv+qw) = p\psi^{\alpha,\beta}(v) + q\psi^{\alpha,\beta}(w)$ for all $v, w \in FCG_C^N$ and all $p, q \in \mathbb{R}_+$, so $\psi^{\alpha,\beta}$is additive on the cone of fuzzy clan games. The stability follows from Theorem 8.8.
(ii) Clearly, each $\psi^{\alpha,\beta}(v) \in C(v)$. Conversely, let $x \in C(v)$. Then, according to Theorem 8.8, $x_i \in [0, D_i v(e^N)]$ for each $i \in N \setminus C$. Hence, for each $i \in \{1, \ldots, m\}$ there is $\alpha_i \in [0,1]$ such that $x_i = \alpha_i D_i v(e^N)$.
Now we show that
$$v(e^N) - \sum_{i=1}^{m} \alpha_i D_i v(e^N) \geq 0. \tag{8.3}$$
Note that $e^C \in \mathcal{F}_{1_C}^N$ is the fuzzy coalition where each non-clan member has participation level 0 and each clan-member has participation level 1. We have
$$\begin{aligned} v(e^N) - v(e^C) &= \sum_{i=1}^{m} \left(v\left(\sum_{k=1}^{i} e^k + e^C\right) - v\left(\sum_{k=1}^{i-1} e^k + e^C\right)\right) \\ &\geq \sum_{i=1}^{m} \left(v\left(e^N\right) - v(e^N - e^i)\right) \geq \sum_{i=1}^{m} D_i v(e^N) \\ &\geq \sum_{i=1}^{m} \alpha_i D_i v(e^N), \end{aligned}$$
where the first inequality follows from the DAMR-property of v by taking $s^1 = \sum_{k=1}^{i} e^k + e^C$, $s^2 = e^N$, $\varepsilon_1 = \varepsilon_2 = 1$, the second inequality follows from Lemma 8.6 with $t = e^N$ and $s_i = 1$, and the third inequality since $D_i v(e^N) \geq 0$ in view of the monotonicity property of v. Hence (8.3) holds.

Inequality (8.3) expresses the fact that the group of clan members is left a non-negative amount in the grand coalition.

The fact that $x_i \geq v(e^i)$ for each $i \in C$ implies that $x_i \geq 0$ for each $i \in \{m+1, \ldots, n\}$. But then there is a vector $\beta \in \Delta(C)$ such that
$$x_i = \beta_i \left(v\left(e^N\right) - \sum_{k=1}^{m} \alpha_k D_k v\left(e^N\right)\right)$$
(take $\beta \in \Delta(C)$ arbitrarily if $v(e^N) - \sum_{i=1}^{m} D_i v(e^N) = 0$, and $\beta_i = x_i \left(v\left(e^N\right) - \sum_{i=1}^m \alpha_i D_i v\left(e^N\right)\right)^{-1}$ for each $i \in C$, otherwise). Hence $x = \psi^{\alpha,\beta}(v)$.

(iii) Trivially, $C(v+w) \supset C(v) + C(w)$ for all $v, w \in FCG_C^N$. Conversely, let $v, w \in FCG_C^N$. Then

$$\begin{aligned} C(v+w) &= \{\psi^{\alpha,\beta}(v+w) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &= \{\psi^{\alpha,\beta}(v) + \psi^{\alpha,\beta}(w) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &\subset \{\psi^{\alpha,\beta}(v) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &\quad + \ \{\psi^{\alpha,\beta}(w) \mid \alpha \in [0,1]^{N\setminus C}, \beta \in \Delta(C)\} \\ &= C(v) + C(w), \end{aligned}$$

where the equalities follow from (ii).

For fuzzy clan games the notion of bi-monotonic participation allocation scheme which we introduce now plays a similar role as pamas for convex fuzzy games (see Section 7.3).

Let $v \in FCG_C^N$. A scheme $(b_{i,t})_{i\in N, t\in \mathcal{F}_{1_C}^N}$ is called a *bi-monotonic participation allocation scheme (bi-pamas)* for v if the following conditions hold:

(i) (*Stability*) $(b_{i,t})_{i\in N} \in C(v_t)$ for each $t \in \mathcal{F}_{1_C}^N$;

(ii) (*Bi-monotonicity w.r.t. participation levels*) For all $s, t \in \mathcal{F}_{1_C}^N$ with $s \le t$ we have:

(ii.1) $s_i^{-1} b_{i,s} \ge t_i^{-1} b_{i,t}$ for each $i \in (N \setminus C) \cap car(s)$;

(ii.2) $b_{i,s} \le b_{i,t}$ for each $i \in C$.

Remark 8.17. The restriction of $(b_{i,t})_{i\in N, t\in \mathcal{F}_{1_C}^N}$ to a crisp environment (where only the crisp coalitions are considered) is a bi-monotonic allocation scheme as studied in Subsection 4.3.2.

Lemma 8.18. *Let $v \in FCG_C^N$. Let $s, t \in \mathcal{F}_{1_C}^N$ with $s \le t$ and let $i \in car(s)$ be a non-clan member. Then $D_i v(s) \ge D_i v(t)$.*

Proof. We have that $D_i v(s) = \lim_{\substack{\varepsilon\to 0 \\ \varepsilon>0}} \varepsilon^{-1}(v(s) - v(s - \varepsilon e^i)) \ge \lim_{\substack{\varepsilon\to 0 \\ \varepsilon>0}} \varepsilon^{-1}(v(t) - v(t - \varepsilon\, e^i)) = D_i v(t)$, where the inequality follows from the DAMR-property of v, with $\varepsilon_1 = \varepsilon_2 = \varepsilon$.

Theorem 8.19. *Let $v \in FCG_C^N$, with $N \setminus C = \{1, \dots, m\}$. Then for each $\alpha \in [0,1]^m$ and $\beta \in \Delta(C) = \Delta(\{m+1, \dots, n\})$ the compensation-sharing rule $\psi^{\alpha,\beta}$ generates a bi-pamas for v, namely $\left(\psi_i^{\alpha,\beta}(v_t)\right)_{i\in N, t\in \mathcal{F}_{1_C}^N}$.*

Proof. We treat only the case $|C| > 1$. In Theorem 8.12(i) we have proved that for each $t \in \mathcal{F}_{1_C}^N$ the Aubin core $C(v_t)$ of the t-restricted game v_t is given by $C(v_t) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(t),\ 0 \le x_i \le t_i D_i v(t)$ for each $i \in N \setminus C$, $0 \le x_i$ for each $i \in C\}$. Then, for each non-clan member i the α-based compensation (regardless of β) in the "grand coalition" t of the t-restricted game v_t is $\psi_i^{\alpha,\beta} = \alpha_i t_i D_i v(t),\ \ i \in \{1, \dots, m\}$. Hence, $\psi_i^{\alpha,\beta} = \beta_i \left(v\,(t) - \sum_{i=1}^m \alpha_i t_i D_i v\,(t)\right)$ for each $i \in \{m+1, \dots, n\}$.

First we prove that for each non-clan member i the compensation per unit of participation level is weakly decreasing when the coalition containing all clan members with full participation level and in which player i is active (i.e. $s_i > 0$) becomes larger.

Let $s, t \in \mathcal{F}_{1_C}^N$ with $s \leq t$ and $i \in car(s) \cap (N \backslash C)$. We have

$$\begin{aligned}\psi_i^{\alpha,\beta}(v_s) &= \alpha_i D_i v_s(e^N) = \alpha_i s_i D_i(v_s) \\ &\geq \alpha_i s_i D_i(v_t) = \alpha_i s_i (t_i)^{-1} D_i v_t(e^N) = s_i (t_i)^{-1} \psi_i^{\alpha,\beta}(v_t),\end{aligned}$$

where the inequality follows from Lemma 8.18 and the second and third equalities by Lemma 8.11. Hence, for each $s, t \in \mathcal{F}_{1_C}^N$ with $s \leq t$ and each non-clan member $i \in car(s)$

$$s_i^{-1} \psi_i^{\alpha,\beta}(v_s) \geq t_i^{-1} \psi_i^{\alpha,\beta}(v_t).$$

Now, denote by $R_\alpha(v_t)$ the α-based remainder for the clan members in the "grand coalition" t of the t-restricted game v_t. Formally,

$$R_\alpha(v_t) = v_t(e^N) - \sum_{i \in N \backslash C} \alpha_i D_i v_t(e^N) = v(t) - \sum_{i \in N \backslash C} \alpha_i D_i v(t).$$

First we prove that for each $s, t \in \mathcal{F}_{1_C}^N$ with $s \leq t$

$$R_\alpha(v_t) \geq R_\alpha(v_s). \tag{8.4}$$

Inequality (8.4) expresses the fact that the remainder for the clan members is weakly larger in larger coalitions (when non-clan members increase their participation level).

Let $s, t \in \mathcal{F}_{1_C}^N$ with $s \leq t$. Then

$$\begin{aligned}v(t) - v(s) &= \sum_{k=1}^{m} \left(v\left(s + \sum_{i=1}^{k} (t_i - s_i) e^i \right) - v\left(s + \sum_{i=1}^{k-1} (t_i - s_i)\, e^i \right) \right) \\ &\geq \sum_{k=1}^{m} (t_k - s_k) D_k v \left(s + \sum_{i=1}^{k} (t_i - s_i)\, e^i \right) \\ &\geq \sum_{k=1}^{m} (t_k - s_k) D_k v(t) \geq \sum_{k=1}^{m} (t_k - s_k) \alpha_k D_k v(t),\end{aligned}$$

where the first inequality follows from Lemma 8.6 and the second inequality from Lemma 8.18. This implies

$$\begin{aligned}v(t) - \sum_{k=1}^{m} t_k \alpha_k D_k v(t) &\geq v(s) - \sum_{k=1}^{m} s_k \alpha_k D_k v(t) \\ &\geq v(s) - \sum_{k=1}^{m} s_k \alpha_k D_k v(s),\end{aligned}$$

where the last inequality follows from Lemma 8.18. So, we proved that $R_\alpha(v_t) \geq R_\alpha(v_s)$ for all $s, t \in \mathcal{F}^N_{1_C}$ with $s \leq t$.

Now note that inequality (8.4) implies that for each clan member the individual share (of the remainder for the whole group of clan members) in v_t, that is $\beta_i R_\alpha(v_t)$, is weakly increasing when non-clan members increase their participation level.

Let $v \in FCG^N_C$ and $x \in C(v)$. Then we call x *bi-pamas extendable* if there exists a bi-pamas $(b_{i,t})_{i\in N, t\in\mathcal{F}^N_{1_C}}$ such that $b_{i,e^N} = x_i$ for each $i \in N$. In the next theorem we show that each core element of a fuzzy clan game is bi-pamas extendable.

Theorem 8.20. *Let $v \in FCG^N_C$ and $x \in C(v)$. Then x is bi-pamas extendable.*

Proof. Let $x \in C(v)$. Then, according to Theorem 8.16(ii), x is of the form $\psi^{\alpha,\beta}(v_{e^N})$. Take now $\left(\psi^{\alpha,\beta}_i(v_t)\right)_{i\in N, t\in\mathcal{F}^N_{1_C}}$, which is a bi-pamas by Theorem 8.19.

Part III

Multichoice games

In a multichoice game each player has a finite number of activity levels to participate with when cooperating with other players. Roughly speaking, cooperative crisp games can be seen as multichoice games where each player has only two activity levels: full participation and not participation at all.

Multichoice games were introduced in [34], [35] and extensively studied also in [18], [20], [47], and [48]. In this part we basically follow [48].

The part is organized as follows. Chapter 9 contains basic notation and notions for multichoice games. In Chapter 10 solution concepts for multichoice games are introduced inspired by classical solution concepts for crisp games. In Chapter 11 balanced and convex multichoice games are presented and special properties of solution concepts on these two classes of games are studied.

9

Preliminaries

Let N be a non-empty finite set of players, usually of the form $\{1, \ldots, n\}$. In a multichoice game each player $i \in N$ has a finite number of activity levels at which he or she can choose to play. In particular, any two players may have different numbers of activity levels. The reward which a group of players can obtain depends on the effort of the cooperating players. This is formalized by supposing that each player $i \in N$ has $m_i + 1$ activity levels at which he can play. We set $M_i := \{0, \ldots, m_i\}$ as the action space of player i, where action 0 means not participating. Elements of $\mathcal{M}^N := \Pi_{i \in N} M_i$ are called (*multichoice*) *coalitions*. The coalition $m = (m_1, \ldots, m_n)$ plays the role of the grand coalition. The empty coalition $(0, \ldots, 0)$ is also denoted by 0. For further use we introduce the notation $M_i^+ := M_i \setminus \{0\}$ and $\mathcal{M}_0^N := \mathcal{M}^N \setminus \{(0, \ldots, 0)\}$. A characteristic function $v : \mathcal{M}^N \to \mathbb{R}$ with $v(0, \ldots, 0) = 0$ gives for each coalition $s = (s_1, \ldots, s_n) \in \mathcal{M}^N$ the worth that the players can obtain when each player i plays at level $s_i \in M_i$.

Definition 9.1. *A* ***multichoice game*** *is a triple (N, m, v) where N is the set of players, $m \in (\mathbb{N} \cup \{0\})^N$ is the vector describing the number of activity levels for all players, and $v : \mathcal{M}^N \to \mathbb{R}$ is the characteristic function.*

If there will be no confusion, we will denote a game (N, m, v) by v. We denote the set of all multichoice games with player set N by MC^N.

Example 9.2. Consider a large building project with a deadline and a penalty for every day this deadline is exceeded. Obviously, the date of completion depends on the effort of all people involved in the project: the greater their effort the sooner the project will be completed. This situation gives rise to a multichoice game. The worth of a coalition where each player works at a certain activity level is defined as minus the penalty that is to be paid given the completion date of the project when every player makes the corresponding effort.

Example 9.3. Suppose we are given a large company with many divisions, where the profits of the company depend on the performance of the divisions. This gives rise to a multichoice game in which the players are the divisions and the worth of a coalition where each division functions at a certain level is the corresponding profit made by the company.

Definition 9.4. *A game $v \in MC^N$ is called **simple** if $v(s) \in \{0,1\}$ for all $s \in \mathcal{M}^N$ and $v(m) = 1$.*

Definition 9.5. *A game $v \in MC^N$ is called **zero-normalized** if no player can gain anything by working alone, i.e. $v\left(je^i\right) = 0$ for all $i \in N$ and $j \in M_i$.*

Definition 9.6. *A game $v \in MC^N$ is called **additive** if the worth of each coalition s equals the sum of the worths of the players when they all work alone at the level they work at in s, i.e. $v(s) = \sum_{i\in N} v\left(s_i e^i\right)$ for all $s \in \mathcal{M}^N$.*

Definition 9.7. *For a game $v \in MC^N$ the **zero-normalization** of v is the game v_0 that is obtained by subtracting from v the additive game a with $a\left(je^i\right) := v\left(je^i\right)$ for all $i \in N$ and $j \in M_i^+$.*

Definition 9.8. *A game $v \in MC^N$ is called **zero-monotonic** if its zero-normalization is monotonic, i.e. $v_0(s) \leq v_0(t)$ for all $s,t \in \mathcal{M}^N$ with $s \leq t$.*

For two sets A and B in the same vector space we set $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$. By convention, the empty sum is zero.

Let $v \in MC^N$. We define $M := \{(i,j) \mid i \in N, j \in M_i\}$ and $M^+ := \left\{(i,j) \mid i \in N, j \in M_i^+\right\}$. A (level) *payoff vector* for the game v is a function $x : M \to \mathbb{R}$, where, for all $i \in N$ and $j \in M_i^+$, x_{ij} denotes the increase in payoff to player i corresponding to a change of activity level $j-1$ to j by this player, and $x_{i0} = 0$ for all $i \in N$.

Let x and y be two payoff vectors for the game v. We say that x is *weakly smaller than* y if for each $s \in \mathcal{M}^N$,

$$X(s) := \sum_{i\in N} X_{is_i} = \sum_{i\in N}\sum_{k=0}^{s_i} x_{ik} \leq \sum_{i\in N}\sum_{k=0}^{s_i} y_{ik} = \sum_{i\in N} Y_{is_i} =: Y(s).$$

Note that this does not imply that $x_{ij} \leq y_{ij}$ for all $i \in N$ and $j \in M_i$. The next example illustrates this point. To simplify the notation in the example we represent a payoff vector $x : M \to \mathbb{R}$ by a deficient matrix $[a_{ij}]$ with $i = 1, \ldots, n$ and $j = 1, \ldots, \max\{m_1, \ldots, m_n\}$. In this matrix $a_{ij} := x_{ij}$ if $i \in N$ and $j \in M_i^+$, and a_{ij} is left out $(*)$ if $i \in N$ and $j > m_i$.

Example 9.9. Let a multichoice game be given with $N = \{1, 2\}$, $m = (2, 1)$ and $v((1, 0)) = v((0, 1)) = 1$, $v((2, 0)) = 2$, $v((1, 1)) = 3$ and $v((2, 1)) = 5$. Consider the two payoff vectors x and y defined by

$$x = \begin{bmatrix} 1 & 2 \\ 2 & * \end{bmatrix}, \quad y = \begin{bmatrix} 2 & 1 \\ 2 & * \end{bmatrix}.$$

Then x is weakly smaller than y, since $X((1, 0)) \leq Y((1, 0))$, $X((1, 1)) \leq Y((1, 1))$ and $X(s) \leq Y(s)$ for all other s. The reason here is that player 1 gets 3 for playing at his second level according to both payoff vectors , while according to y player 1 gets 2 for playing at his first level and according to x player 1 gets only 1 at the first level.

10

Solution concepts for multichoice games

In this chapter we present the extension of solution concepts for cooperative crisp games to multichoice games. Special attention is paid to imputations, cores and stable sets, and to solution concepts based on the marginal vectors of a multichoice game (Shapley values and the Weber set).

10.1 Imputations, cores and stable sets

Let $v \in MC^N$. A payoff vector $x : M \to \mathbb{R}$ is called *efficient* if $X(m) = \sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m)$ and it is called *level increase rational* if, for all $i \in N$ and $j \in M_i^+$, x_{ij} is at least the increase in worth that player i can obtain when he works alone and changes his activity from level $j-1$ to level j, i.e. $x_{ij} \geq v\left(je^i\right) - v\left((j-1)\,e^i\right)$.

Definition 10.1. *Let $v \in MC^N$. A payoff vector $x : M \to \mathbb{R}$ is an **imputation** of v if it is efficient and level increase rational.*

We denote the set of imputations of a game $v \in MC^N$ by $I(v)$. It can be easily seen that

$$I(v) \neq \emptyset \Leftrightarrow \sum_{i \in N} v\left(m_i e^i\right) \leq v(m). \tag{10.1}$$

Definition 10.2. *The **core** $C(v)$ of a game $v \in MC^N$ consists of all $x \in I(v)$ that satisfy $X(s) \geq v(s)$ for all $s \in \mathcal{M}^N$.*

Let $s \in \mathcal{M}_0^N$ and $x, y \in I(v)$. The imputation y dominates the imputation x via coalition s, denoted by $y \operatorname{dom}_s x$, if $Y(s) \leq v(s)$ and $Y_{is_i} > X_{is_i}$ for all $i \in car(s)$. We say that the imputation y dominates the imputation x if there exists $s \in \mathcal{M}_0^N$ such that $y \operatorname{dom}_s x$.

Definition 10.3. *The **dominance core** $DC(v)$ of a game $v \in MC^N$ consists of all $x \in I(v)$ for which there exists no y such that y dominates x.*

In Theorems 10.4, 10.6 and 10.7 we deal with the relations between the core and the dominance core of a multichoice game.

Theorem 10.4. *For each game $v \in MC^N$, we have $C(v) \subset DC(v)$.*

Proof. Let $x \in C(v)$ and suppose $y \in I(v)$ and $s \in \mathcal{M}_0^N$, such that $y \operatorname{dom}_s x$. Then

$$v(s) \geq Y(s) = \sum_{i \in N} Y_{is_i} > \sum_{i \in N} X_{is_i} = X(s) \geq v(s),$$

which clearly gives a contradiction. Therefore, x is not dominated.

Let $v \in MC^N$ be a zero-normalized game (cf. Definition 9.7) and x a payoff vector for v. Then the condition of level increase rationality boils down to the condition $x \geq 0$. For an additive game a we have $C(a) = DC(a) = I(a) = \{x\}$, where $x : M \to \mathbb{R}$ is the payoff vector with $x_{ij} := a\left(je^i\right) - a\left((j-1)\, e^i\right)$ for all $i \in N$ and $j \in M_i^+$. Now we have the following

Proposition 10.5. *Let $v \in MC^N$ and v_0 be the zero-normalization of v. Let x be a payoff vector for v. Define $y : M \to \mathbb{R}$ by $y_{ij} := x_{ij} - v\left(je^i\right) + v\left((j-1)\, e^i\right)$ for all $i \in N$ and $j \in M_i^+$. Then we have*
(i) $x \in I(v) \Leftrightarrow y \in I(v_0)$,
(ii) $x \in C(v) \Leftrightarrow y \in C(v_0)$,
(iii) $x \in DC(v) \Leftrightarrow y \in DC(v_0)$.

We leave the proof of this proposition as an exercise to the reader.

Theorem 10.6. *Let $v \in MC^N$ with $DC(v) \neq \emptyset$. Then $C(v) = DC(v)$ if and only if the zero-normalization v_0 of v satisfies $v_0(s) \leq v_0(m)$ for all $s \in \mathcal{M}^N$.*

Proof. By Proposition 10.5 it suffices to prove this theorem for zero-normalized games. So, suppose v is zero-normalized. Further, suppose $C(v) = DC(v)$ and let $x \in C(v)$. Then

$$v(m) = X(m) = \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} + \sum_{i \in N} \sum_{j=s_i+1}^{m_i} x_{ij} \geq v(s)$$

for all $s \in \mathcal{M}^N$.

Now suppose $v(s) \leq v(m)$ for all $s \in \mathcal{M}^N$. Since $C(v) \subset DC(v)$ (cf. Theorem 10.4), it suffices to prove that $x \notin DC(v)$ for all $x \in I(v) \setminus C(v)$. Let $x \in I(v) \setminus C(v)$ and $s \in \mathcal{M}_0^N$ such that $X(s) < v(s)$. Define $y : M^+ \to \mathbb{R}$ as follows

$$y_{ij} := \begin{cases} x_{ij} + \frac{v(s)-X(s)}{\sum_{k \in N} s_k} & \text{if } i \in N \text{ and } j \in \{1, \ldots, s_i\}\,, \\ \frac{v(m)-v(s)}{\sum_{k \in N}(m_k - s_k)} & \text{if } i \in N \text{ and } j \in \{s_i + 1, \ldots, m_i\}\,. \end{cases}$$

It follows readily from the definition of y that y is efficient. Since $x \geq 0$, $v(s) > X(s)$ and $v(m) \geq v(s)$, it follows that $y \geq 0$. Hence, y is also level increase rational and we conclude that $y \in I(v)$.

For $i \in N$ and $j \in \{1, \ldots, s_i\}$ we have that $y_{ij} > x_{ij}$. Hence, $Y_{is_i} > X_{is_i}$ for all $i \in N$. This and the fact that

$$Y(s) = X(s) + \sum_{i \in N} \sum_{j=1}^{s_i} \frac{v(s) - X(s)}{\sum_{k \in N} s_k} = v(s)$$

imply that $y \operatorname{dom}_s x$. Hence, $x \notin DC(v)$.

Using Theorems 10.4 and 10.6 we can easily prove Theorem 10.7. Note that this theorem also holds for cooperative crisp games, because the class of multichoice games contains the class of cooperative crisp games.

Theorem 10.7. *Let $v \in MC^N$ with $C(v) \neq \emptyset$. Then $C(v) = DC(v)$.*

Proof. It suffices to prove the theorem for zero-normalized games (cf. Proposition 10.5). So, suppose that v is zero-normalized. From the first part of the proof of Theorem 10.6 we see that the fact that $C(v) \neq \emptyset$ implies that $v(s) \leq v(m)$ for all $s \in \mathcal{M}^N$. Because $C(v) \subset DC(v)$ (cf. Theorem 10.4), we know that $DC(v) \neq \emptyset$. Now Theorem 10.6 immediately implies $C(v) = DC(v)$.

Considering Theorem 10.7 one might ask oneself if there actually exist games where the core is not equal to the dominance core. The answer to this question is given in Example 10.8, where we provide a multichoice game with an empty core and a non-empty dominance core.

Example 10.8. Let a multichoice game be given with $N = \{1, 2\}$, $m = (2, 1)$ and $v\left((1,0)\right) = v\left((0,1)\right) = 0$, $v\left((2,0)\right) = \frac{1}{4}$ and $v\left((1,1)\right) = v\left((2,1)\right) = 1$. An imputation x should satisfy the following (in)equalities:

$$x_{11} + x_{12} + x_{21} = 1,\ x_{11} \geq 0,\ x_{21} \geq 0,\ x_{12} \geq \frac{1}{4}.$$

Hence, we obtain

$$I(v) = co\left\{ \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{3}{4} & * \end{bmatrix}, \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & * \end{bmatrix} \right\}.$$

Note that for this game an imputation can only dominate another imputation via the coalition $(1,1)$ and, since $x_{11}+x_{21} \leq \frac{3}{4}$ for all $x \in I(v)$, this gives us

$$DC(v) = co\left\{\begin{bmatrix} 0 & \frac{1}{4} \\ \frac{3}{4} & * \end{bmatrix}, \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ 0 & * \end{bmatrix}\right\}.$$

Finally, for none of the elements x of the dominance core $x_{11}+x_{21} \geq v((1,1))$. Since $C(v) \subset DC(v)$ one obtains $C(v)=\emptyset$. Note that for the zero-normalization v_0 of v it holds that $v_0((1,1)) = 1 > \frac{3}{4} = v_0((2,1))$.

We leave it to the reader to find an example of a cooperative crisp game for which the core is not equal to the dominance core (a game with three players will suffice).

For the game in Example 10.8 both the core and the dominance core are convex sets. This is generally true, as it is stated next.

Theorem 10.9. *Let $v \in MC^N$. Then the following two assertions hold:*
(i) $C(v)$ is convex,
(ii) $DC(v)$ is convex.

Proof. We omit the proof of part (i) because it is a simple exercise. In order to prove part (ii) it suffices to prove that $DC(v)$ is convex if v is zero-normalized. So, suppose that v is zero-normalized. Obviously, if $DC(v)=\emptyset$, then it is convex. Now suppose $DC(v) \neq \emptyset$. We define a game $w \in MC^N$ by $w(s) := \min\{v(s), v(m)\}$ for all $s \in \mathcal{M}^N$. It can be easily seen that

$$w(m) = v(m). \tag{10.2}$$

We show that $DC(v) = DC(w) = C(w)$. Since $DC(v) \neq \emptyset$, we know that $I(v) \neq \emptyset$. Since v is zero-normalized, this implies $v(m) \geq 0$ (cf. (10.1)) and

$$w\left(je^i\right) = \min\left\{v\left(je^i\right), v(m)\right\} = 0 \tag{10.3}$$

for all $i \in N$ and $j \in M_i$.

Using (10.2) and (10.3) we see that $I(w) = I(v)$.

Now let $s \in \mathcal{M}_0^N$ and let $x, y \in I(v) = I(w)$. Since $w(s) \leq v(s)$ we see that if $x \operatorname{dom}_s y$ in w, then $x \operatorname{dom}_s y$ in v. On the other hand, if $x \operatorname{dom}_s y$ in v, then $X(s) \leq v(s)$ and

$$X(s) = \sum_{i\in N}\sum_{j=1}^{m_i} x_{ij} - \sum_{i\in N}\sum_{j=s_i+1}^{m_i} x_{ij} \leq v(m)$$

and therefore $X(s) \leq w(s)$ and $x \operatorname{dom}_s y$ in w.

We conclude that

$$DC(w) = DC(v). \tag{10.4}$$

This implies that $DC(w) \neq \emptyset$. Since w is zero-normalized (cf. (10.3)) and

$$w(s) = \min\{v(s), v(m)\} \leq v(m) = w(m),$$

by Theorem 10.6,

$$C(w) = DC(w). \tag{10.5}$$

Now (10.4), (10.5) and part (i) of this theorem immediately imply that $DC(v)$ is convex.

Other sets of payoff vectors for multichoice games which are based on the notion of domination are introduced in [47] as follows.

Let $v \in MC^N$ and $2^{I(v)} := \{A \mid A \subset I(v)\}$. We introduce two maps, $D : 2^{I(v)} \to 2^{I(v)}$ and $U : 2^{I(v)} \to 2^{I(v)}$, given for all $A \subset I(v)$ by

$$D(A) := \{x \in I(v) \mid \text{there exists } a \in A \text{ that dominates } x\};$$
$$U(A) := I(v) \setminus D(A).$$

The set $D(A)$ consists of all imputations that are dominated by some element of A. The set $U(A)$ consists of all imputations that are undominated by elements of A. Hence, $DC(v) = U(I(v))$.

A set $A \subset I(v)$ is *internally stable* if elements of A do not dominate each other, i.e. $A \cap D(A) = \emptyset$, and it is *externally stable* if all imputations not in A are dominated by an imputation in A, i.e. $I(v) \setminus A \subset D(A)$. A set $A \subset I(v)$ is a *stable set* (cf. [45]) if it is both internally and externally stable.

It can be easily seen that a set $A \subset I(v)$ is stable if and only if A is a fixed point of U, i.e. $U(A) = A$. The following theorem is an extension towards multichoice games of Theorem 2.11.

Theorem 10.10. *Let $v \in MC^N$. Then the following two assertions hold:*
(i) Every stable set contains the dominance core as a subset;
(ii) If the dominance core is a stable set, then there are no other stable sets.

It has been shown in [39] that there exist cooperative crisp games without a stable set. Therefore, since all our definitions are consistent with the corresponding definitions for cooperative crisp games, we may conclude that multichoice games do not always have a stable set.

10.2 Marginal vectors, Shapley values and the Weber set

Let $v \in MC^N$. Suppose the grand coalition $m = (m_1, \ldots, m_n)$ forms step by step, starting from the coalition $(0, \ldots, 0)$ and where in each step the level of one of the players is increased by 1. So, in particular, there are $\sum_{i \in N} m_i$ steps in this procedure. Now assign for every player to each level the marginal value that is created when the player reaches that particular level from the level directly below. This is formalized as follows.

An *admissible ordering* (for v) is a bijection $\sigma : M^+ \to \left\{1, \ldots, \sum_{i \in N} m_i\right\}$ satisfying $\sigma((i,j)) < \sigma((i,j+1))$ for all $i \in N$ and $j \in \{1, \ldots, m_i - 1\}$. The number of admissible orderings for v is $\frac{\left(\sum_{i \in N} m_i\right)!}{\Pi_{i \in N}(m_i!)}$. The set of all admissible orderings for a game v will be denoted by $\Xi(v)$.

Now let $\sigma \in \Xi(v)$ and let $k \in \left\{1, \ldots, \sum_{i \in N} m_i\right\}$. The coalition that is present after k steps according to σ, denoted by $s^{\sigma,k}$, is given by

$$s_i^{\sigma,k} := \max\{j \in M_i \mid \sigma((i,j)) \leq k\} \cup \{0\}$$

for all $i \in N$, and the *marginal vector* $w^\sigma : M \to \mathbb{R}$ corresponding to σ is defined by

$$w_{ij}^{\sigma} := v\left(s^{\sigma,\sigma((i,j))}\right) - v\left(s^{\sigma,\sigma((i,j))-1}\right)$$

for all $i \in N$ and $j \in M_i^+$.

In general the marginal vectors of a multichoice game are not necessarily imputations, but for zero-monotonic games they are.

Theorem 10.11. *Let $v \in MC^N$ be zero-monotonic. Then for every $\sigma \in \Xi(v)$ the marginal vector corresponding to σ is an imputation of v.*

We can consider the average of the marginal vectors of a multichoice game that will give us an extension of the Shapley value for crisp games to multichoice games.

Definition 10.12. *([48]) Let $v \in MC^N$. Then the* ***Shapley value*** *$\Phi(v)$ is the average of all marginal vectors of v, in formula*

$$\Phi_i(v) := \frac{\Pi_{i \in N}(m_i!)}{\left(\sum_{i \in N} m_i\right)!} \sum_{\sigma \in \Xi(v)} w^\sigma.$$

It turns out that there is more than one reasonable extension of the definition of the Shapley value for cooperative crisp games to multichoice games. Following [47] we will consider the Shapley values that were introduced in [34]. These values were defined by using weights on the actions, thereby extending ideas of *weighted Shapley values* (cf. [37]).

We start by introducing the notion of a *minimal effort game* that is the analogue of a crisp unanimity game for multichoice games. A minimal effort game $u_s \in MC^N$ with $s \in \mathcal{M}_0^N$ is defined by

$$u_s(t) := \begin{cases} 1 \text{ if } t_i \geq s_i \text{ for all } i \in N, \\ 0 \text{ otherwise,} \end{cases}$$

for all $t \in \mathcal{M}^N$. The name of these games is clear: all players have to put in a minimal effort in order to obtain profit.

The definition of dividends for crisp games (cf. [31]) can be extended to multichoice games as follows: for $v \in MC^N$

$$\begin{aligned} \triangle_v(0) &:= 0 \\ \triangle_v(s) &:= v(s) - \sum_{t \leq s, t \neq s} \triangle_v(t). \end{aligned} \tag{10.6}$$

Theorem 10.13. *The minimal effort games $u_s \in MC^N$, $s \in \mathcal{M}_0^N$, form a basis of the space MC^N. Moreover, for $v \in MC^N$ it holds that*

$$v = \sum_{s \in \mathcal{M}_0^N} \triangle_v(s) u_s.$$

When introducing the values of [35], we must restrict ourselves to multichoice games where all players have the same number of activity levels. So, let MC_*^N denote the subclass of MC^N with the property that $m_i = m_j$ for all $i, j \in N$. For $v \in MC_*^N$ set $\widetilde{m} := m_i$ ($i \in N$ arbitrarily) and let for each $j \in \{0, \ldots, \widetilde{m}\}$ a weight $w_j \in \mathbb{R}$ be associated with level j such that higher levels have larger weights, i.e. $0 = w_0 < w_1 < \ldots < w_{\widetilde{m}}$. The value Ψ is defined with respect to the weights w.

Definition 10.14. *([35]) For $s \in \mathcal{M}_0^N$, the value $\Psi^w(u_s)$ of the minimal effort game u_s is given by*

$$\Psi_{ij}^w(u_s) = \begin{cases} \frac{w_j}{\sum_{i \in N} w_{s_i}} & \text{if } j = s_i, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N$ and $j \in M_i$.

Further, the value $\Psi^w(v)$ of an arbitrary game $v \in MC_*^N$ is determined by

$$\Psi^w(v) := \sum_{s \in \mathcal{M}_0^N} \triangle_v(s) \Psi^w(u_s).$$

An axiomatic characterization of this value has been provided in [35], using additivity, the carrier property, the minimal effort property and a fourth axiom that explicitly uses weights. We describe these properties of an allocation rule $\gamma : MC^N \to \mathbb{R}^{M^+}$ below.

- *Additivity:* For all $v, w \in MC^N$

$$\gamma(v + w) = \gamma(v) + \gamma(w).$$

- *Carrier property:* If t is a *carrier* of $v \in MC^N$, i.e. $v(s) = v(s \wedge t)$ for all $s \in \mathcal{M}^N$, then

$$\sum_{i \in car(t)} \sum_{j=1}^{t_i} \gamma_{ij}(v) = v(m).$$

- *Minimal effort property:* If $v \in MC^N$ and $t \in \mathcal{M}^N$ are such that $v(s) = 0$ for all s with $s \ngeq t$, then for all $i \in N$ and $j < t_i$

$$\gamma_{ij}(v) = 0.$$

- *Weight property:* Suppose that the weights $0 = w_0 < w_1 < \ldots < w_{\widetilde{m}}$ are given. If a game $v \in MC_*^N$ is a multiple of a minimal effort game, say $v = \beta u_s$, $s \in \mathcal{M}^N$, then for all $i, j \in N$

$$\gamma_{i,s_i}(v).w_{s_j} = \gamma_{j,s_j}(v).w_{s_i}.$$

The reader is referred to [35] for the proof of the following

Theorem 10.15. *Consider the class MC_*^N. Let weights $0 = w_0 < \ldots < w_{\widetilde{m}}$ be given. Then Ψ^w is the unique allocation rule on MC_*^N satisfying additivity, the carrier property, the minimal effort property and the weight property.*

The value Φ introduced in Definition 10.12 can be characterized by additivity, the carrier property and the hierarchical strength property, which in fact incorporates the minimal effort property and the weight property, where the difference lies in the fact that the 'weights' that are used are now determined by the numbers of the activity levels of the players (cf. [28]).

- The *hierarchical strength* $h_s((i,j))$ in $s \in \mathcal{M}_0^N$ of $(i,j) \in M^+$ with $j \leq s_i$ is defined by the average number of $\sigma \in \Xi(v)$ in which (i,j) is s-maximal, i.e. $h_s((i,j))$ equals

$$\frac{\Pi_{i \in N}(m_i!)}{\left(\sum_{i \in N} m_i\right)!} \left| \left\{ \sigma \in \Xi(v) \mid \sigma((i,j)) = \max_{(k,l):l \leq s_k} \sigma((k,l)) \right\} \right|.$$

- An allocation rule $\gamma : MC^N \to \mathbb{R}^{M^+}$ satisfies *the hierarchical strength property* if for each $v \in MC^N$ which is a multiple of a minimal effort game, say $v = \beta u_s$ with $s \in \mathcal{M}_0^N$ and $\beta \in \mathbb{R}$, we have that for all $(i_1, j_1), (i_2, j_2) \in M^+$

$$\gamma_{i_1,j_1}(v).h_s(i_2, j_2) = \gamma_{i_2,j_2}(v).h_s(i_1, j_1).$$

The reader is referred to [28] for the proof of the following

Theorem 10.16. *The value Φ is the unique allocation rule on MC_*^N satisfying additivity, the carrier property and the hierarchical strength property.*

An interesting question that arises now is whether the value Φ is related to the values Ψ^w. We provide an example of a multichoice game for which the value Φ is not equal to any of the values Ψ^w.

Example 10.17. Let $v \in MC^{\{1,2\}}$ with $m = (3,3)$ and let $v = u_{(1,2)} + u_{(3,1)} + u_{(2,3)}$. There are 20 admissible orderings for this game. Some calculation shows that

$$\Phi(v) = \begin{bmatrix} \frac{4}{20} & \frac{4}{20} & \frac{19}{20} \\ \frac{1}{20} & \frac{16}{20} & \frac{16}{20} \end{bmatrix}.$$

Now, suppose we have weights $w_1 < w_2 < w_3$ associated with the activity levels. Then the corresponding value Ψ^w is

$$\Psi^w(v) = \begin{bmatrix} \frac{w_1}{w_1+w_2} & \frac{w_2}{w_2+w_3} & \frac{w_3}{w_1+w_3} \\ \frac{w_1}{w_1+w_3} & \frac{w_2}{w_1+w_2} & \frac{w_3}{w_2+w_3} \end{bmatrix}.$$

Hence, if we want to find weights w such that $\Psi^w(v) = \Phi(v)$, then these weights should satisfy the conditions $0 < w_1 < w_2 < w_3$, $w_2 = 4w_1$, $w_3 = 4w_2$ and $w_3 = 19w_1$. Clearly, it is impossible to find weights satisfying all these conditions.

Instead of concentrating of the average of the marginal vectors of a multichoice game, one can also consider the convex hull of the marginal vectors of a multichoice game, i.e. its Weber set.

Definition 10.18. *([48]) The* ***Weber set*** *$W(v)$ of a game $v \in MC^N$ is defined as*

$$W(v) := co\,\{w^\sigma \mid \sigma \in \Xi(v)\}.$$

The next theorem shows a relation between the core $C(v)$ and the Weber set $W(v)$ of a multichoice game v.

Theorem 10.19. *Let $v \in MC^N$ and $x \in C(v)$. Then there is a $y \in W(v)$ that is weakly smaller than x.*

Proof. It will be actually proved that for each game $v \in MC^N$ and each $x \in \widetilde{C}(v)$ there is a vector $y \in W(v)$ such that y is weakly smaller than x, where $\widetilde{C}(v)$ is a core catcher of $C(v)$ $\left(C(v) \subset \widetilde{C}(v)\right)$ given by

$$\left\{x \in I(v) \mid X(s) \geq v(s)\ \forall s \in \mathcal{M}^N, x_{i0} = 0\ \forall i \in N\right\}.$$

We will do so by induction on the number of levels involved in the game v. Two basic steps can be distinguished.

(i) Let $v \in MC^{\{1\}}$ with $m_1 \in \mathbb{N}$ being arbitrary. Then there is only one marginal vector y, which satisfies

$$y_{1j} = v\left(je^1\right) - v\left((j-1)\,e^1\right)$$

for all $j \in \{1, \ldots, m_1\}$. Suppose $x \in \widetilde{C}(v)$. Then

$$X\left(m_1 e^1\right) = v\left(m_1 e^1\right) = Y\left(m_1 e^1\right)$$

and

$$X\left(je^1\right) \geq v\left(je^1\right) = Y\left(je^1\right) \text{ for all } j \in \{1,\ldots,m_1\}.$$

Hence, y is weakly smaller than x.

(ii) Let $v \in MC^{\{1,2\}}$ with $m = (1,1)$. Then there are two marginal vectors,

$$y^1 = \begin{bmatrix} v\left(e^1\right) \\ v\left(e^1+e^2\right) - v\left(e^1\right) \end{bmatrix} \text{ and } y^2 = \begin{bmatrix} v\left(e^1+e^2\right) - v\left(e^2\right) \\ v\left(e^2\right) \end{bmatrix}.$$

Suppose $x \in \widetilde{C}(v)$. Then

$$x_{11} \geq v\left(e^1\right),\, x_{21} \geq v\left(e^2\right) \text{ and } x_{11} + x_{21} = v\left(e^1+e^2\right).$$

Hence, x is a convex combination of y^1 and y^2. We conclude that $x \in W(v)$.

(iii) Now let $v \in MC^N$ be such that $|\{i \in N \mid m_i > 0\}| \geq 2$ and $\sum_{i\in N} m_i > 2$. Suppose we already proved the statement for all games $\overline{v} \in MC^{\overline{N}}$ with $\sum_{i\in\overline{N}} \overline{m}_i < \sum_{i\in N} m_i$. Since, obviously, $\widetilde{C}(v)$ and $W(v)$ are both convex sets, it suffices to prove that for all extreme points x of $\widetilde{C}(v)$ we can find $y \in W(v)$ such that y is weakly smaller than x. So, let x be an extreme point of $\widetilde{C}(v)$. Then let $t \in \mathcal{M}^N$ be such that $1 \leq \sum_{i\in N} t_i \leq \sum_{i\in N} m_i - 1$ and $X(t) = v(t)$. The game v can be split up into a game u with vector of activity levels t and a game w with vector of activity levels $m - t$, defined by

$$u(s) := v(s) \text{ for all } s \in \mathcal{M}^N \text{ with } s \leq t$$

and

$$w(s) := v(s+t) - v(t) \text{ for all } s \in \mathcal{M}^N \text{ with } s \leq m - t.$$

The payoff x can be also split up into two parts,

$$x^u : \{(i,j) \mid i \in N, j \in \{0,\ldots,t_i\}\} \to \mathbb{R}$$

and

$$x^w : \{(i,j) \mid i \in N, j \in \{0,\ldots,m_i - t_i\}\} \to \mathbb{R}$$

defined by

$$x^u_{ij} := x_{ij} \text{ for all } i \in N \text{ and } j \in \{0,\ldots,t_i\}$$

and

$$x^w_{ij} := \begin{cases} x_{i,j+t_i} & \text{if } i \in N \text{ and } j \in \{1,\ldots,m_i - t_i\}, \\ 0 & \text{if } i \in N \text{ and } j = 0. \end{cases}$$

Now $x^u \in \widetilde{C}(u)$ because $X^u(t) = X(t) = v(t) = u(t)$ and $X^u(s) = X(s) \geq v(s) = u(s)$ for all $s \in \mathcal{M}^N$ with $s \leq t$. Further, $x^w \in \widetilde{C}(w)$ because

$$\begin{aligned} X^w(m-t) &= \sum_{i\in N}\sum_{j=1}^{m_i-t_i} x_{i,j+t_i} \\ &= X(m) - X(t) = v(m) - v(t) = w(m-t) \end{aligned}$$

and

$$\begin{aligned} X^w(s) &= \sum_{i\in N}\sum_{j=1}^{s_i} x_{i,j+t_i} \\ &= X(s+t) - X(t) \geq v(s+t) - v(t) = w(s) \end{aligned}$$

for all $s \in \mathcal{M}^N$ with $s \leq m-t$.

Using the induction hypothesis, one can find $y^u \in W(u)$ such that y^u is weakly smaller than x^u, and one can find $y^w \in W(w)$ such that y^w is weakly smaller than x^w. Then $y := (y^u, y^w)$ is weakly smaller than $x := (x^u, x^w)$. Hence, the only thing to prove still is that $y \in W(v)$.

For the payoff vector

$$z^1 : \{(i,j) \mid i \in N, j \in \{0,\ldots,t_i\}\} \to \mathbb{R}$$

for u and the payoff vector

$$z^2 : \{(i,j) \mid i \in N, j \in \{0,\ldots,m_i - t_i\}\} \to \mathbb{R}$$

for w one defines the payoff vector $\left(z^1, z^2\right) : M \to \mathbb{R}$ for v as follows:

$$\left(z^1, z^2\right)_{ij} := \begin{cases} z^1_{ij} \text{ if } i \in N \text{ and } j \in \{0,\ldots,t_i\}\,, \\ z^2_{ij} \text{ if } i \in N \text{ and } j \in \{t_i+1,\ldots,m_i\}\,. \end{cases}$$

We prove that

$$(W(u), W(w)) := \left\{\left(z^1, z^2\right) \mid z^1 \in W(u), z^2 \in W(w)\right\}$$

is a subset of $W(v)$. Note that $(W(u), W(w))$ and $W(v)$ are convex sets. Hence, it suffices to prove that the extreme points of $(W(u), W(w))$ are elements of $W(v)$. Suppose $\left(z^1, z^2\right)$ is an extreme point of $(W(u), W(w))$. Then, obviously, z^1 is a marginal vector of u and z^2 is a marginal vector of w. Let $\sigma \in \Xi(u)$ and $\rho \in \Xi(w)$ be such that z^1 is the marginal vector of u corresponding to σ and z^2 is the marginal vector of w corresponding to ρ. Then $\left(z^1, z^2\right)$ is the marginal vector of v corresponding to the admissible ordering τ for v defined by $\tau\left((i,j)\right) := \sigma\left((i,j)\right)$ if $i \in N$ and $j \in \{1,\ldots,t_i\}$, and $\tau\left((i,j)\right) := \rho\left((i, j-t_i)\right) + \sum_{i\in N} t_i$ if $i \in N$ and $j \in \{t_i+1,\ldots,m_i\}$.

Hence, $\left(z^1, z^2\right) \in W(v)$ and this completes the proof.

11

Classes of multichoice games

11.1 Balanced multichoice games

In [48] a notion of balancedness for multichoice games is introduced and a theorem in the spirit of Theorem 2.4 is proved, which we present in the following.

Definition 11.1. *A game $v \in MC^N$ is called* ***balanced*** *if for all maps $\lambda : \mathcal{M}_0^N \to \mathbb{R}_+$ satisfying*

$$\sum_{s \in \mathcal{M}_0^N} \lambda(s) e^{car(s)} = e^N \tag{11.1}$$

it holds that $\sum_{s \in \mathcal{M}_0^N} \lambda(s) v_0(s) \leq v_0(m)$, where v_0 is the zero-normalization of v.

Note that this definition coincides with the familiar definition of balancedness for cooperative crisp games $v \in MC^N$ with $m = (1, \ldots, 1)$ (cf. Definition 1.17).

The next theorem is an extension to multichoice games of a theorem proved in [9] and [59] which gives a necessary and sufficient condition for the nonemptiness of the core of a game.

Theorem 11.2. *Let $v \in MC^N$. Then $C(v) \neq \emptyset$ if and only if v is balanced.*

Proof. It suffices to prove the theorem for zero-normalized games.

Suppose v is zero-normalized, $C(v) \neq \emptyset$ and $x \in C(v)$. Then we define a payoff vector $y : M^+ \to \mathbb{R}$ by

$$y_{ij} := \begin{cases} 0 & \text{if } i \in N \text{ and } j \in \{2, \ldots, m_i\}, \\ \sum_{l=1}^{m_i} x_{il} & \text{if } i \in N \text{ and } j = 1. \end{cases}$$

Then, obviously, $y \in C(v)$. Further, one can identify y with the vector $(y_{11}, \ldots, y_{n1})$. This proves that $C(v) \neq \emptyset$ if and only if there exist $z_1, \ldots, z_n \in \mathbb{R}_+$ such that

$$\sum_{i \in N} z_i = v(m) \tag{11.2}$$

and

$$\sum_{i \in car(s)} z_i \geq v(s) \tag{11.3}$$

for all $s \in \mathcal{M}_0^N$.

Obviously, there exist $z_1, \ldots, z_n \in \mathbb{R}_+$ satisfying (11.2) and (11.3) if and only if for all $i \in N$ and all $s \in \mathcal{M}_0^N$ we have

$$v(m) = \min \left\{ \sum_{i \in N} z_i \mid z_i \in \mathbb{R}, \sum_{i \in car(s)} z_i \geq v(s) \right\}. \tag{11.4}$$

From the duality theorem of linear programing theory (cf. Theorem 1.31) we know that (11.4) is equivalent to

$$v(m) = \max \left\{ \sum_{s \in \mathcal{M}_0^N} \lambda(s) v(s) \mid (11.1) \text{ holds and } \lambda(s) \geq 0 \right\}. \tag{11.5}$$

It can be easily seen that (11.5) is equivalent to v being balanced.

The rest of this section deals with multichoice flow games arising from flow situations with committee control and their relations with balanced multichoice games. Our presentation of the results is according to [48]. Using multichoice games to model flow situations with committee control allows one to require a coalition to make a certain effort in order to be allowed to use the corresponding arcs, for example to do a necessary amount of maintenance of the used arcs. Flow situations with committee control generate either crisp flow games when the control games on the arcs are crisp games or they generate multichoice games when the control games on the arcs are multichoice games. For an introduction to crisp flow games we refer the reader to [38].

Let N be a set of players and let $m \in (\mathbb{N} \cup \{0\})^N$. A flow situation consists of a directed network with two special nodes called the source and the sink. For each arc there are a capacity constraint and a constraint with respect to the allowance to use that arc. If l is an arc in the network and w is the (simple) control game for arc l, then a coalition s is allowed to use arc l if and only if $w(s) = 1$. The capacity of an arc l in the network is denoted by $c_l \in (0, \infty)$. The flow game corresponding to a flow situation assigns to a coalition s the maximal flow that coalition s can send through the network from the source to the sink.

For cooperative crisp games it was shown in [38] that a nonnegative cooperative crisp game is totally balanced if and only if it is a flow game corresponding to a flow situation in which all arcs are controlled by a single player (cf. Theorem 4.4). The corresponding definitions of a dictatorial simple game and of a totally balanced game for the multichoice case are given below.

Definition 11.3. *A simple game $v \in MC^N$ is called **dictatorial** if there exist $i \in N$ and $j \in M_i^+$ such that $v(s) = 1$ if and only if $s_i \geq j$ for all $s \in \mathcal{M}_0^N$.*

Definition 11.4. *A game $v \in MC^N$ is called **totally balanced** if for every $s \in \mathcal{M}_0^N$ the subgame v_s is balanced, where $v_s(t) := v(t)$ for all $t \in \mathcal{M}^N$ with $t \leq s$.*

However, as exemplified in [48], one cannot generalize Theorem 4.4 to multichoice games. In order to reach balancedness, we will restrict ourselves to zero-normalized games. Then we have the following

Theorem 11.5. *Consider a flow situation in which all control games are zero-normalized and balanced. Then the corresponding flow game $v \in MC^N$ is non-negative, zero-normalized and balanced.*

Proof. It is obvious that v is zero-normalized and non-negative. Now, in order to prove that v is balanced, let $L = \{l_1, \ldots, l_p\}$ be a set of arcs with capacities $c_1, \ldots, c_p$ and control games $w_1, \ldots, w_p$ such that every directed path from the source to the sink contains an arc in L and the capacity of L, $\sum_{r=1}^{p} c_r$, is minimal. From a theorem in [27] we find that $v(m) = \sum_{r=1}^{p} c_r$ and $v(s) \leq \sum_{r=1}^{p} c_r w_r(s)$ for all $s \in \mathcal{M}^N$.

Now, let $x^r \in C(w_r)$ for all $r \in \{1, \ldots, p\}$. Define $y := \sum_{r=1}^{p} c_r x^r$. Then

$$Y(m) = \sum_{r=1}^{p} c_r X^r(m) = \sum_{r=1}^{p} c_r w_r(m) = v(m) \tag{11.6}$$

and

$$Y(s) = \sum_{r=1}^{p} c_r X^r(s) \geq \sum_{r=1}^{p} c_r w_r(s) \geq v(m), \tag{11.7}$$

for all $s \in \mathcal{M}^N$.

Now, let $i \in N$ and $j \in M_i^+$. Since $c_r \geq 0$ and $x_{ij}^r \geq 0$ for all $r \in \{1, \ldots, p\}$ it easily follows that

$$y_{ij} = \sum_{r=1}^{p} c_r x_{ij}^r \geq 0. \tag{11.8}$$

Now (11.6), (11.7) and (11.8) imply $y \in C(v)$. Hence, v is balanced.

We can prove the converse of Theorem 11.5 using

Theorem 11.6. *Each non-negative zero-normalized balanced multichoice game is a non-negative linear combination of zero-normalized balanced simple games.*

Proof. Let $v \in MC^N$ be non-negative, zero-normalized and balanced. We provide an algorithm to write v as a non-negative linear combination of zero-normalized balanced simple games.

Suppose $v \neq 0$ and let $x \in C(v)$. Let $k \in N$ be the smallest integer in

$$\{i \in N \mid \exists j \in N \text{ s.t. } x_{ij} > 0\}$$

and let l be the smallest integer in $\left\{j \in M_k^+ \mid x_{kj} > 0\right\}$.

Further, let

$$\beta := \min\left\{x_{kl}, \min\left\{v(s) \mid s \in \mathcal{M}_0^N, s_k \geq l, v(s) > 0\right\}\right\}$$

and let w be defined by

$$w(s) := \begin{cases} 1 \text{ if } s_k \geq l \text{ and } v(s) > 0, \\ 0 \text{ otherwise,} \end{cases}$$

for each $s \in \mathcal{M}^N$. Then w is a zero-normalized balanced simple game and $\beta > 0$.

Let $\overline{v} := v - \beta w$ and let $\overline{x} : M \to \mathbb{R}$ be defined by

$$\overline{x}_{ij} := \begin{cases} x_{kl} - \beta \text{ if } i = k \text{ and } j = l, \\ x_{ij} \qquad \text{otherwise.} \end{cases}$$

Note that $\overline{v}$ is a non-negative zero-normalized game, $v = \overline{v} + \beta w$, and $\overline{x} \in C(\overline{v})$.

Further,

$$|\{(i,j) \in M\} \mid \overline{x}_{ij} > 0| < |\{(i,j) \in M\} \mid x_{ij} > 0|$$

or

$$\left|\left\{s \in \mathcal{M}^N \mid \overline{v}(s) > 0\right\}\right| < \left|\left\{s \in \mathcal{M}^N \mid v(s) > 0\right\}\right|.$$

If $\overline{v} \neq 0$ we follow the same procedure with $\overline{v}$ in the role of v and $\overline{x}$ in the role of x. It can be easily seen that if we keep on repeating this procedure, then after finitely many steps we will obtain the zero game. Suppose this happens after q steps. Then we have found $\beta_1, \ldots, \beta_q > 0$ and zero-normalized balanced simple games $w_1, \ldots, w_q$ such that $v = \sum_{r=1}^{q} \beta_r w_r$.

Theorem 11.7. *Let $v \in MC^N$ be non-negative, zero-normalized and balanced. Then v is a flow game corresponding to a flow situation in which all control games are zero-normalized and balanced.*

Proof. According to Theorem 11.6 we can find $k \in \mathbb{N}$, $\beta_1, \ldots, \beta_k > 0$ and zero-normalized balanced games $w_1, \ldots, w_k$ such that $v = \sum_{r=1}^{k} \beta_r w_r$.

Consider now a flow situation with k arcs, where for each $r \in \{1, \ldots, k\}$ the capacity restriction of arc l_r is given by β_r and the control game of l_r is w_r. It can be easily seen that the flow game corresponding to the described flow situation is the game v.

Combining Theorems 11.5 and 11.7 we obtain

Corollary 11.8. *Let $v \in MC^N$ be non-negative and zero-normalized. Then v is balanced if and only if v is a flow game corresponding to a flow situation in which all control games are zero-normalized and balanced.*

11.2 Convex multichoice games

A game $v \in MC^N$ is called *convex* if

$$v\left(s \wedge t\right) + v\left(s \vee t\right) \geq v(s) + v(t) \tag{11.9}$$

for all $s, t \in \mathcal{M}^N$. Here $\left(s \wedge t\right)_i := \min\left\{s_i, t_i\right\}$ and $\left(s \vee t\right)_i := \max\left\{s_i, t_i\right\}$ for all $i \in N$.

For a convex game $v \in MC^N$ it holds that

$$v\left(s + t\right) - v(s) \geq v\left(\overline{s} + t\right) - v\left(\overline{s}\right) \tag{11.10}$$

for all $s, \overline{s}, t \in \mathcal{M}^N$ satisfying $\overline{s} \leq s$, $\overline{s}_i = s_i$ for all $i \in car(t)$ and $s + t \in \mathcal{M}^N$. This can be obtained by putting s and $\overline{s} + t$ in the roles of s and t, respectively, in expression (11.9). In fact, every game satisfying expression (11.10) is convex.

In the following we denote the class of convex multichoice games with player set N by CMC^N. For these games we can say more about the relation between the core and the Weber set.

Theorem 11.9. *Let $v \in CMC^N$. Then $W(v) \subset C(v)$.*

Proof. Note that convexity of both $C(v)$ and $W(v)$ implies that it suffices to prove that $w^\sigma \in C(v)$ for all $\sigma \in \Xi(v)$. So, let $\sigma \in \Xi(v)$. Efficiency of w^σ follows immediately from the definition of this game. That w^σ is level increase rational follows straightfordwardly when we use expression (11.10). Now let $s \in \mathcal{M}^N$. The ordering σ induces an admissible ordering $\sigma' : \left\{(i,j) \mid i \in N, j \in \{1, \ldots, s_i\}\right\} \to \left\{1, \ldots, \sum_{i \in N} s_i\right\}$ in an obvious way.

Since $s^{\sigma',\sigma'((i,j))} \leq s^{\sigma,\sigma((i,j))}$ for all $i \in N$ and $j \in \{1,\ldots,s_i\}$, the convexity of v implies $w_{ij}^{\sigma'} \leq w_{ij}^{\sigma}$ for all $i \in N$ and $j \in \{1,\ldots,s_i\}$. Hence,

$$\sum_{i\in N}\sum_{j=0}^{s_i} w_{ij}^{\sigma} \geq \sum_{i\in N}\sum_{j=0}^{s_i} w_{ij}^{\sigma'} = v(s).$$

We conclude that $w^{\sigma} \in C(v)$.

In contrast with convex crisp games for which $C(v) = W(v)$ holds (cf. Theorem 4.9(v)), the converse of Theorem 11.9 is not true for convex multichoice games. We provide an example of a game $v \in CMC^N$ with $W(v) \subset C(v)$, $W(v) \neq C(v)$.

Example 11.10. Let $v \in CMC^{\{1,2\}}$ with $m = (2,1)$ and $v((1,0)) = v((2,0)) = v((0,1)) = 0$, $v((1,1)) = 2$ and $v((2,1)) = 3$. There are three marginal vectors,

$$w_1 = \begin{bmatrix} 0 & 0 \\ 3 & * \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 2 & * \end{bmatrix}, w_3 = \begin{bmatrix} 2 & 1 \\ 0 & * \end{bmatrix}.$$

Some calculation shows that $C(v) = co\{w_1, w_2, w_3, x\}$, where $x = \begin{bmatrix} 3 & 0 \\ 0 & * \end{bmatrix}$. We see that $x \notin co\{w_1, w_2, w_3\} = W(v)$.

The core element x in Example 11.10 seems to be too large: note that w_3 is weakly smaller than x and w_3 is still in the core $C(v)$. This inspires the following

Definition 11.11. *For a game $v \in MC^N$ the* ***set*** $C_{\min}(v)$ ***of minimal core elements*** *is defined as follows*

$$\{x \in C(v) \mid \nexists y \in C(v) \text{ s.t. } y \neq x \text{ and } y \text{ is weakly smaller than } x\}.$$

Now we can formulate

Theorem 11.12. *Let $v \in CMC^N$. Then $W(v) = co(C_{\min}(v))$.*

Proof. We start by proving that all marginal vectors are minimal core elements. Let $\sigma \in \Xi(v)$. Then $w^{\sigma} \in C(v)$ (cf. Theorem 11.9). Suppose $y \in C(v)$ is such that $y \neq w^{\sigma}$ and y is weakly smaller than w^{σ}. Let $i \in N$ and $j \in M_i^+$ be such that $Y(je^i) < \sum_{l=1}^{j} w_{il}^{\sigma}$ and consider $t := s^{\sigma,\sigma((i,j))}$. Then

$$Y(t) = \sum_{k\in N} Y\left(t_k e^k\right) < \sum_{k\in N}\sum_{l=0}^{t_k} w_{kl}^{\sigma} = v(t), \tag{11.11}$$

where the inequality follows from the fact that $t_i = j$ and the last equality follows from the definitions of t and w^{σ}. Now (11.11) implies that $y \notin C(v)$. Hence, we see that $w^{\sigma} \in C_{\min}(v)$. This immediately implies that

$$W(v) \subset co\,(C_{\min}(v))\,. \tag{11.12}$$

Now let x be a minimal core element. We prove that $x \in W(v)$. According to Theorem 10.19 we can find a payoff vector $y \in W(v)$ that is weakly smaller than x. Using (11.12) we see that $y \in co\,(C_{\min}(v)) \subset C(v)$. Since x is minimal we may conclude that $x = y \in W(v)$. Hence, $W(v) = co\,(C_{\min}(v))$.

Note that Theorem 11.12 implies that for a convex crisp game the core coincides with the Weber set. The converse of Theorem 11.12 also holds, as shown in

Theorem 11.13. *Let $v \in MC^N$ with $W(v) = co\,(C_{\min}(v))$. Then $v \in CMC^N$.*

Proof. Let $s, t \in \mathcal{M}^N$. Clearly, there is an order σ that is admissible for v and that has the property that there exist k, l with $0 \leq k \leq l \leq \sum_{i\in N} m_i$ such that $s \wedge t = s^{\sigma,k}$ and $s \vee t = s^{\sigma,l}$. Note that for the corresponding marginal vector w^σ we have that $w^\sigma \in co\,(C_{\min}(v)) \subset C(v)$. Using this we see

$$\begin{aligned} v(s) + v(t) &\leq \sum_{i\in N}\sum_{j=1}^{s_i} w_{ij}^\sigma + \sum_{i\in N}\sum_{j=1}^{t_i} w_{ij}^\sigma \\ &= \sum_{i\in N}\sum_{j=1}^{(s\wedge t)_i} w_{ij}^\sigma + \sum_{i\in N}\sum_{j=1}^{(s\vee t)_i} w_{ij}^\sigma \\ &= v\,(s \wedge t) + v\,(s \vee t)\,, \end{aligned}$$

where the last equality follows from the definition of w^σ. Hence, v is convex.

From Theorems 11.12 and 11.13 we immediately obtain

Corollary 11.14. *Let $v \in MC^N$. Then $v \in CMC^N$ if and only if $W(v) = co\,(C_{\min}(v))$.*

With respect to stable sets of convex multichoice games we have the next result.

Theorem 11.15. *Let $v \in CMC^N$. Then $C(v)$ is the unique stable set.*

Proof. Using Corollary 11.14 we see that $C(v) \neq \emptyset$. Hence, it follows from Theorem 10.7 that $C(v) = DC(v)$. So, by Theorem 10.10(ii) we know that it suffices to prove that $C(v)$ is a stable set.

Internal stability of $C(v)$ is obvious. To show external stability, let $x \in I(v) \setminus C(v)$. We construct $z \in C(v)$ that dominates x. First we choose $s \in \mathcal{M}_0^N$ such that

$$|car(s)|^{-1}\,(v(s) - X(s)) = \max_{t\in\mathcal{M}_0^N} |car(t)|^{-1}\,(v(t) - X(t))\,.$$

Since $x \notin C(v)$ it holds that

$$\left|car(s)^{-1}\right| (v(s) - X(s)) > 0. \tag{11.13}$$

Now, let σ be an order that is admissible for v with the property that there exists k such that $s = s^{\sigma,k}$. Then (cf. Theorem 11.9) the corresponding marginal vector w^σ is an element of $C(v)$ and, moreover, it holds that $\sum_{i\in N}\sum_{j=1}^{s_i} w_{ij}^\sigma = v(s)$. For notational convenience we set $y := w^\sigma$. We define the payoff vector z by $z_{ij} = x_{ij}$ if $i \in car(s)$ and $2 \le j \le s_i$, $z_{ij} = x_{i1} + |car(s)|^{-1}(v(s) - X(s))$ if $i \in car(s)$ and $j = 1$, $z_{ij} = y_{ij}$ if $i \notin car(s)$ or $i \in car(s)$ and $j > s_i$; $z_{i0} = 0$.

Using the fact that $x, y \in I(v)$ and recalling (11.13), it can be easily seen that z is level increase rational. Further, $Z(m) = X(s) + (v(s) - X(s)) + (Y(m) - Y(s)) = v(s) + (v(m) - v(s)) = v(m)$, where the second equality follows from the way we choose y. This shows that z is also efficient and, hence, $z \in I(v)$. Since $Z(s) = v(s)$ and $Z_{is_i} = X_{is_i} + |car(s)|^{-1}(v(s) - X(s)) > X_{is_i}$ for all $i \in car(s)$, it holds that $z\,\mathrm{dom}_s\,x$.

The only thing that is left to prove is $z \in C(v)$. So, let $t \in \mathcal{M}_0^N$. We distinguish two cases.

(a) If $car(t) \cap car(s) = \emptyset$, then $Z(t) = Y(t) \ge v(t)$ since $y \in C(v)$.

(b) If $car(t) \cap car(s) \ne \emptyset$, then

$$\begin{aligned} Z(s\wedge t) &= (Z - X)(s\wedge t) + X(s\wedge t) \\ &= |car(s)\cap car(t)|\,.\,|car(s)|^{-1}(v(s) - X(s)) + X(s\wedge t) \\ &\ge v(s\wedge t) - X(s\wedge t) + X(s\wedge t) = v(s\wedge t), \end{aligned} \tag{11.14}$$

where the inequality follows from (11.13). Hence,

$$\begin{aligned} Z(t) &= \sum_{i\in N}\sum_{j=1}^{s_i\wedge t_i} z_{ij} + \sum_{i\in N: t_i > s_i}\ \sum_{j=s_i+1}^{t_i} y_{ij} \\ &= Z(s\wedge t) + \sum_{i\in N}\sum_{j=1}^{s_i\vee t_i} y_{ij} - \sum_{i\in N: s_i\ge t_i}\sum_{j=1}^{s_i} y_{ij} - \sum_{i\in N: s_i< t_i}\sum_{j=1}^{s_i} y_{ij} \\ &= Z(s\wedge t) + Y(s\vee t) - Y(s) \\ &\ge v(s\wedge t) + v(s\vee t) - v(s), \end{aligned} \tag{11.15}$$

where the last equality follows from (11.14) and the fact that $y \in C(v)$ is such that $Y(s) = v(s)$. Using convexity of v, we see that the last expression in (11.15) is larger or equal of $v(t)$. This completes the proof of the theorem.

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