A discussion of the applications of fuzzy sets to game theory

by

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ABSTRACT

Fuzzy Game Theory is a growing field in mathematics, economics, and computer science. In this paper, we follow the thread of fuzzy logic to the doorstep of social sciences and game theory. We then examine how fuzzy sets have been applied equilibrium theory of games and discuss some cases where Nash equilibria can be show to exist, and ultimately a situation where a Nash equilibrium does not arise unless the game is one of perfect information.

Preface

The pure mathematician knows that pure mathematics has an

end in itself which is more allied with philosophy.

–Philip Jourdain, in the introduction to Georg Cantor's

Contributions to the Founding of the Theory of Transfinite Numbers

I am interested in the mathematics which can describe interactions between humans and groups of humans in the political, sociological, and economics spheres (or social sciences in general). In contemporary political science, game theory (called rational choice theory) is king. The assumptions of rational choice models are, briefly, that human interactions can be modeled by players, whose choices can be modeled by the game, who have a preference ordering concerning the outcome of those choices, and that players will act in a way to maximize the utility of the outcome given their behavior. All utility functions (aka payoff function) from the set of outcomes to the utilities of those outcomes which satisfy the players preference functions are allowed (Myerson (1997)). A major strength of game theoretical analysis of social science situations is the flexibility that this model is afforded by allowing agents to order their preferences, and by giving preferences an order, but not necessarily a particular value. Although like much of mathematics, the foundations of game theory can be stretched ever further into the past. But the major source of the field was John von Neumann, and Morgenstern and Von Neumann (1944) marked the arrival of the subject. It has since been a major contribution in areas from economics (John Nash, Thomas Schelling) to ethical theories (John Rawls, Amartya Sen).

In 1965, Loft Zadeh introduced (Zadeh (1965)) the world to the term fuzzy, as a formalization of vagueness. The field has been applied with a good deal of success to engineering, with fuzzy control systems able to do things like cook rice or shift gears in a car with great efficiency. Since fuzzy set operators extend the idea of the logical and and or to non-dichotomous systems, behavioral scientists have asked if there is a way to use fuzzy mathematical models to analyze social systems. The book by Smithson addresses exactly that question. Suggestions about how this could be done can be found in the computer science field of natural language processing (Dubois and Prade (1980)).

In modern social science, game theory is the method of making mathematical models. Fuzzy Logic gives a new tool which on the face seems to apply to this type of modeling as well. Mares (2001) mixes the concepts in his book discussing coalition games with fuzzy pay-offs (fuzzy expected utility functions), developing the ideas of fuzzy core, fuzzy balancedness, and fuzzy shapely values. Others such as Smithson and Verkuilen (2006) and Ragin (2000) examine using fuzzy sets in linear statistical models. Mansur (1995) looks at the application of fuzzy sets to some introductory concepts of microeconomics.

1 Vagueness

This, of course, is the answer to the old puzzle about the man who went bald. It is supposed that at first he was not bald, that he lost his hairs one by one, and that in the end he was bald; therefore, it is argued, there must have been one hair the loss of which converted him into a bald man. This, of course, is absurd. Baldness is a vague conception; some men are certainly bald, some are certainly not bald, while between them there are men of whom it is not true to say they must be either be bald or not bald. The law of excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as, in fact, all symbols are.

-B. Russell *Vagueness, 1923*

The ship wherein Theseus and the youth of Athens returned [from Crete] had thirty oars, and was preserved by the Athenians down even to the time of Demetrius Phalereus, for they took away the old planks as they decayed, putting in new and stronger timber in their place, insomuch that this ship became a standing example among the philosophers, for the logical question of things that grow; one side holding that the ship remained the same, and the other contending that it was not the same.

-Plutarch

What is more, there cannot be anything between two contradictories, but of any one subject, one thing must either be asserted or denied. This is clear if we first define what is truth and what is falsehood. A falsity is a statement of that which is that it is not, or of that which is not that it is; and a truth is a statement of that which is that it is, or of that which is not that it is not. Hence, he who states of anything that it is, or that it is not, will either speak truly or speak falsely. But of what is neither being nor nonbeing it is not said that it is or that it is not.

-Aristotle *Metaphysics IV*

There is a sense in which human languages are vague. Logic, however, is usually presented as being precise. In the fourth century B.C. Socratic philosopher Eubulides of Miletus discussed the paradox of the bald man. And Plutarch reported the paradox of the ship of Theseus as coming from Greek legend. But classic logic, even as formed by Aristotle held that a statement must be either true or false. A man must be bald or not, a ship must be that sailed by Thesues or a different ship altogether. This *law of excluded middle* left us with a logic which was unable to give expression to the vaguarities of human language and the vaguarities of human categorization. Passing over the middle age stoic philosophers who largely followed in Aristotle's footsteps, it is important to make mention of Italian Renaissance philosopher of language, Lorenzo Valla (1407-1457), who felt that each hair made some difference. John Locke (1632-1704) and Gottfried Leibniz (1646-1716) discussed the question of what sorts of things formed a natural kind, that is which have a boundary defined by natural law. Although the pair disagreed on the extent to which boundaries between different things are formed by the human mind, both agree that borderline cases are a matter of opinion. Alexander Bain (1818-1903) addressed the question in his *Logic of Relatives* (1870), concluding that 'a certain margin must be allowed as indetermined, and open to difference of opinion.' Williamson (1996)

In the late nineteenth and early twentieth century, the concept of vagueness was reexamined by various scientists and philosophers. A full account exists in Williamson (1996) of the historical backdrop of fuzzy theory. Early on, the concept of vagueness was mentioned in order to exclude it. Gottlob Frege's (1848-1925) concern that without sharp borders logic rules would be broken were included in his *Grundgesetze der Arithmetik* (1893-1903) influenced Bertrand Russell (1872-1970) as well as Polish philosophers Tadeusz Kotarbinski (1886-1981) and Kazimierz Ajdukiewicz (1890-1963), all of whom tried to give precise definitions of vagueness. Charles Sanders Peirce (1839-1914) was working independently of Frege gave a definition of vague in a philosophical dictionary he wrote in 1901. British philosopher Max Black (1909-1988) based his discussions of vagueness largely on examples of "'borderline cases"'. Kotarbinski began using gradedness of truths in writings as early as 1923. Physician Ludwik Fleck (1896-1961) applied the Poles' ideas to medical diagnosis.

Karl Menger (1902-1985), formerly of the Vienna circle, generalized metric spaces toward probabilistic concepts, introducing the concept of a triangular norm (T-Norm) in 1942. Menger suggested in 1951 that an item need not be absolutely an element of a set, but rather there could be a mapping from the element to the probability that the element is in the set. He was also influenced in a 1951 paper by Henri Poincare (1854-1912) claim that equality is not transitive in the physical reality, as when we say two objects in the real world are equal, we mean that they are indistinguishable, but that if one follows a chain of indistinguishable items, one need not say that the first and the last item remain indistinguishable. Menger acknowledged the fuzzy sets of Lotfi Zadeh (1921-), but did not explore the differences between probability sets and fuzzy sets in his work. In the early 1960s, Richard E. Bellman (1920-1984), Robert Kalaba (1926-2004), and Zadeh began exploring the concept of fuzzy sets at RAND Corporation, leading to a joint memorandum in 1964 and a paper in the journal *Systems Theory* entitled *Fuzzy Sets and Systems* in 1965 (Zadeh (1965)).

Now we will depart the historical development of fuzzy sets and fuzzy logic in order that we can build our theories from the bottom. In this thesis, we will leave open the question of whether vagueness comes only from linguistic (or human) causes.

2 Fuzzy Sets and Fuzzy Logic

To understand the mathematical foundations of fuzzy sets and fuzzy logic, we will start our study with a review of the concepts of universal algebra. This will give us a foundation from which we can understand the generalizability of fuzzyness, before we look at specific applications. We will then look at how universal algebra serves as a basis for traditional logic, and then show how those definitions generalize to fuzzy logic through the concept of fuzzy sets. We will then look at some of the variations of fuzzy sets, of which we will tend to use the most simple so that its application can be most easily understood.

2.1 Universal Algebra

An n-ary relation ρ on the sets $A_1, \ldots, A_n$ is specified by giving an ordered n-tuple $(a_1, \ldots, a_n)$ of elements with each element $a_i \in A_i$ such that the n-tuple is in the relation ρ. In this way, the relation ρ is a subset of the product set $A_1 \times \cdots \times A_n$. The number n is called the arity of the relation. We often say unary instead of 1-ary, binary instead of 2-ary and tertiary instead of 3-ary. For binary relations, we may often write "$a_1 \rho a_2$" while for relations of any arity may be written "$(a_1, \ldots, a_n) \in \rho$" or "$\rho(a_1, \ldots, a_n)$".

A binary relation, ρ, on a set X is complete if $a\rho B$ or $b\rho a$ for every a, $b \in X$, A relation is transitive if $a\rho b$ and $b\rho c$ implies $a\rho c$ for every a, b, $c \in X$. A relation is reflexive if $a\rho a$ for every $a \in X$. a *preference relation* is a complete, transitive, reflexive binary relation. A preference relation, $\precsim$, on X is continuous if for all k given sequences $(a^k)_i$ and $(b^k)_i$ in X that converge to $a \in X$ and $b \in X$ respectively such that $a^k \precsim b^k$, then $a \precsim b$. A preference relation $\precsim$, on $\mathbb{R}$ is *quasiconcave* if for every $b \in \mathbb{R}$ the set $\{a \in \mathbb{R} : a \precsim b\}$ is convex. Recall, a set is convex if given two points in the set, a line segment connecting those points is also in the set.

A function $f : A_1 \longrightarrow A_2$ is a binary relation such that for each $a_1 \in A_1$ there is exactly one $a_2 \in A_2$ such that $(a_1, a_2) \in f)$, and is often written "‘$f(a_1) = a_2$"’. A function $f(a_1, \ldots, a_n)$ "‘of n

variables"' where $a_i \in A_i$ for all $i \in \{1, \ldots, n\}$ is a function $f : A_1 \times \cdots \times A_n \longrightarrow B$. For each $a_i \in A_i$ for $i \in \{1, \ldots, n\}$, $(a_1 \times \cdots \times a_n) \in A_1 \times \cdots \times A_n$ and $f((a_1 \times \cdots \times a_n)) \in B$. We may omit one set of brackets from this final expression to simplify our notation.

Definition 2.1.1. An n-ary *operation* on the set A is a function $f : A^n \longrightarrow A$.

Definition 2.1.2. An *algebra* $\mathbf{A}$ is a pair $\langle A, F\rangle$ with A a nonempty set and $F = \langle f_i : i \in I\rangle$ a sequence of operations on A, where I is some index set. We call A the *universe* or the *underlying set* of $\mathbf{A}$, and the f_i are the *fundamental* or *basic* operations of the algebra. The *similarity type* ρ is a function $\rho : I \longrightarrow \omega$ such that each $i \in I$ is assigned to the arity of f_i from the fundamental operations of the algebra.

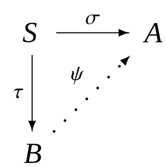
Often, an algebra $\langle A, F\rangle$ is written $\langle A, f_1, \ldots, f_n\rangle$, and the operations are written in descending order of their arity. So if we denote the order of the domain of a function f as $|f|$, then $|f_1| \geq \cdots \geq |f_n|$. With this notation, we can see that $\rho(f) = |f|$. The similarity type is often written simply as a sequence of arities, i.e. $\langle |f_1|, \ldots, |f_n|\rangle$.

Definition 2.1.3. A *subalgebra* of an algebra $\mathbf{A} = \langle A, F\rangle$ is a subset $\mathbf{B} = \langle B, G\rangle$ of $\mathbf{A}$ if $\mathbf{B}$ forms a algebra with the operations of $\mathbf{A}$ restricted to the set B, i.e., if for all $i \in I$ and for all $g_i \in G$, we have $g_i = f_i \restriction_{B^{\rho(i)}}$.

Equality between two algebras $\mathbf{A} = \langle A, F\rangle$ and $\mathbf{B} = \langle B, F\rangle$ requires $A = B$ and $f_A = f_B$ for all $f \in F$. If two algebras have at least the same similarity type, they are called similar. Any intersection of subalgebras is clearly a subalgebra. So, given any subset B of A, there is a unique smallest subalgebra containing B, namely, the subalgebra $\cap\{S : S$ subalgebra of A, $S \supset B\}$. This is the subset generated by B and is denoted by $\langle B\rangle_T$ or occasionally $\langle B\rangle$.

Definition 2.1.4. A *homomorphism* between algebras $\mathbf{A} = \langle A, F\rangle$ and $\mathbf{B} = \langle B, G\rangle$ is a function $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ such that for all $i \in I$ the index set for the operations of A, and all $a_1, \ldots, a_n \in A$, we have: $\phi(f_i(a_1, \ldots, a_n)) = g_i(\phi(a_1), \ldots, \phi(a_n))$ for $f_i \in F$ and $g_i \in G$. That is to sat "ϕ" preserves all the operations of A. Clearly the composition of two homomorphisms is a homomorphism. If $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ is an invertible homomorphism, then $\phi^{-1} : \mathbf{A} \longrightarrow \mathbf{B}$ is also a homomorphism. In this case we call ϕ an isomorphism, and we say that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

Definition 2.1.5. Let S be any set, let $\mathbf{A}$ be a algebra of type ρ, and let $\sigma : S \longrightarrow \mathbf{A}$ be a function. we say that $(\mathbf{A}, \sigma)$ (or informally just $\mathbf{A}$) is a "free algebra" (of type ρ) on the set S of "free generators" if, for every algebra $\mathbf{B}$ of type ρ and function $\tau : S \longrightarrow \mathbf{B}$, there exists a unique homomorphism $\psi : \mathbf{B} \longrightarrow \mathbf{A}$ such that $\psi\sigma = \tau$.



Note that for any set S and any similarity type ρ, there exists a free algebra of type ρ generated by S (Sometimes said to be a free algebra on S), and that this free algebra is unique up to isomorphism.

For the following definition, let A be some algebra, while F is the algebra on the set $X_n = \{x_1, \ldots, x_n\}$. Now for any elements $a_1, \ldots, a_n \in A$ there exists a unique homomorphism $\phi : F \longrightarrow A$ with $\phi(x_i) = a_i$ for $i = 1, \ldots, n$. If $w \in F$, then $\phi(W) \in A$, and w is uniquely determined by $a_1, \ldots, a_n$. Thus we can define a function $w_A(a_1, \ldots, a_n) := \phi(w)$, (note: we may omit the subscript A). In particular, if we take $A = F$ and $a_i = x_i$ for $i \in 1, \ldots, n$, then ϕ is the identity and $w(x_1, \ldots, x_n) = w$.

Definition 2.1.6. A T-*word* in *variables* $x_1, \ldots, x_n$ is an element of the free algebra of type ρ on the set $X_n = \{x_1, \ldots, x_n\}$ of free generators. A *word* in the elements $a_1, \ldots, a_n$ of an algebra A of type ρ is an element $w(a_1, \ldots, a_n) \in A$, where w is a T-word in the variables $x_1, \ldots, x_n$. A *algebra variable* is an element of the free generating set of a free algebra.

Definition 2.1.7. Any algebra with a similarity type $\rho = \langle 2, 0\rangle$ is a *propositional algebra*. We may generally write the algebra as $\langle X, \Rightarrow, F\rangle$. The *propositional algebra* $P(X)$ *of the propositional calculus on the set* X *of propositional variables* is the free propositional algebra of type ρ on X.

Here we may bridge between what we have been doing and what is easily recognizable to be logic. In this algebra, X are all free variables. the nullary operation F is often called FALSE and may also be represented by the integer 0. The binary operation $\Rightarrow$ is sometimes called implication. The operations $\neg$ (NOT), $\vee$ (AND), $\wedge$ (OR), and $\Leftrightarrow$ (IFF) can be defined in terms of these two operations as follows:

This determines the form of our algebra of propositions. In ordinary usage, we are interested in the truth or falsity of a statement, and first the truth and falsity of elements of the propositional algebra

$$\neg p := p \Rightarrow F$$
$$p \vee q := (\neg p) \Rightarrow q$$
$$p \wedge q := \neg(\neg p \vee \neg q)$$
$$p \Leftrightarrow q := (p \Rightarrow q) \wedge (q \Rightarrow p).$$

Table 2.1 operations $\neg$, $\vee$, $\wedge$, and $\Leftrightarrow$

$P(X)$. In traditional two-valued logic, we may consider functions (called valuations) which assign to each $p \in P(X)$ one of two values, 1, or 0 (TRUE or FALSE).

Definition 2.1.8. A *valuation* of $P(X)$ is a proposition algebra homomorphism $v : P(X) \longrightarrow \{0, 1\}$. Thus we may say that $p \in P(X)$ is *true with respect to* v if $v(p) = 1$ and that $p \in P(X)$ is *false with respect to* v if $v(p) = 0$.

2.2 Introduction to Fuzzy Set Theory

Conceptually we can think of a set as being fuzzy when its elements belong only partly to it. Thus a *fuzzy set* A is given by first specifying a universe of elements X which are to be discussed and a scale of *truth degrees* L and a rule which associates with each element x of X a value l from L which represents the degree to which x belongs to the fuzzy set A.

So which of these is the fuzzy set? None, in fact, but let's begin again. First we need to define a *residuated lattice*:

Definition 2.2.1. A residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1, \rangle$ of type $\langle 2, 2, 2, 2, 0, 0, \rangle$ where

i. 0 is the least element and 1 is the greatest element,

ii. $\langle L, \otimes, 1, \rangle$ is a commutative monoid, $\otimes$ is associative, commutative, and the identity $x \otimes 1 = x$ holds.

iii. the *adjointness property* holds, i.e. $x \leq y \rightarrow z \Leftrightarrow x \otimes y \leq z$ holds $\forall x, y, z \in L$ with $\leq$ the lattice ordering in L.

We will later call $\otimes$ a t-norm and $\rightarrow$ implication.

A residuated lattice is called *complete* if $\mathbf{L} = \langle L, \wedge, \vee, 0, 1, \rangle$ is a complete lattice, that is if every subset S of L has both a greatest lower bound and a least upper bound. These are denoted by: $\bigwedge S$ (meet) and $\bigvee S$ (join).

Taking a big step ahead, a fuzzy set is a mapping $A : X \to \mathbf{L}$ for X a set and $\mathbf{L}$ a residuated lattice. A complete residuated lattice with $L = \{0, 1\}$ is the same propositional algebra as before. In using a generalized residuated lattice, operations like meet (AND), join (OR), and implication can be generalized. This generalization allows us to talk about fuzzy logic, which will be introduced in the following section. We will later mention *types* of fuzzy sets to discuss an extension of the concept where the underlying set of the residuated lattice is itself fuzzy. Another way of denoting a fuzzy set is as a pair, one from the universal set, and one from the residuated lattice. In some cases, we may also talk about a fuzzy set based upon some crisp set, X, such that for each subset of X, we want to define a degree of belonging. We will denote this by, $\mathcal{L}(X) \in (\mathcal{P}(X), \mathbf{L})$, where $\mathcal{P}(X)$ is the powerset of X.

2.3 fuzzy logic

The subject of fuzzy logic generalizes traditional logic. In depth reviews of the algebraic basis of fuzzy logic include Gerla (2001), Cignoli et al. (1999), Nguyen (1999), and Hjek (2001) In all forms of logic, our main objective is often to discover whether or not a statement (or formula) is true.

In classic logic, we take the evaluation of a formula of 0 to mean that the formula is false, while an evaluation of 1 means that the formula is true. We then evaluate our connectives $\vee$, $\neg$, $\wedge$, and $\to$ according to the following tables:

$\neg$	
0	1
1	0

Table 2.2 NOT operation

$\vee$	0	1
0	0	1
1	1	1

Table 2.3 OR operation

$\wedge$	0	1
0	0	0
1	0	1

Table 2.4 AND operation

$\to$	0	1
0	1	0
1	1	1

Table 2.5 implication

To talk about a logic, it is important to talk about the connectives between the elements. We can compare fuzzy logic to classical logic if we consider t-norms, t-conorms, and fuzzy implication to be analogous to AND, OR and classic implication. For simplicity, we will use the set $[0, 1]$ with the usual lattice ordering as the image of our fuzzy mappings rather than a more general residuated lattice.

Definition 2.3.1. A *T-norm* is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ with the following properties:

i. Commutativity: $T(a, b) = T(b, a)$

ii. Monotonicity: $T(a, b) \le T(c, d)$ if $a \le c$ and $b \le d$

iii. Associativity: $T(a, T(b, c)) = T(T(a, b), c)$

iv. 0 is the null element: $T(a, 0) = 0$

v. 1 is the identity element: $T(a, 1) = a$

Definition 2.3.2. A t-norm is called *Archimedean* if 0 and 1 are its only idempotent elements.

Definition 2.3.3. An Archimedean t-norm is called *strict* if 0 is its only nilpotent element.

Notice that a T-norm generalizes a triangle norm in a metric space in that $T(a, b) \le T(a, 1) + T(b, 1)$ for all $a, b \in [0, 1]$.

It can be seen how the T-norm conforms to our idea of an AND operator, since monotonicity requires a conjointer with a ’less true’ proposition to be ’less true.’ Associativity and commutativity are included, and the last two properties state that truth values 0 and 1 correspond to ’false’ and ’true,’ respectively.

Just as OR in traditional logic is in some sense dual to AND, T-conorms are dual to T-norms.

Definition 2.3.4. A *T-conorm* can be defined by $\perp(a, b) = \neg T(\neg a, \neg b)$

This, of course, generalizes De Morgan’s laws.

It follows that a T-conorm satisfies dual properties to T-norms, namely:

i. Commutativity: $\perp(a, b) = \perp(b, a)$

ii. Monotonicity: $\perp(a, b) \le; \perp(c, d)$ if $a \le; c$ and $b \le; d$;

iii. Associativity: $\perp(a, \perp(b, c)) = \perp(\perp(a, b), c)$;

iv. Null element: $\bot(a, 1) = 1$

v. Identity element: $\bot(a, 0) = a$

The T-conorm is used to represent intersection in fuzzy set theory.

There are different fuzzy logics that have been developed using different T-norms and T-conorms. The following T-norms and T-conorms are often used:

$$\top_{\text{min}}(a,b) = \min\{a,b\} \qquad \bot_{\text{max}}(a,b) = \max\{a,b\}$$

$$\top_{\text{Luka}}(a,b) = \max\{0, a+b-1\} \qquad \bot_{\text{Luka}}(a,b) = \min\{a+b, 1\}$$

$$\top_{\text{prod}}(a,b) = a \cdot b \qquad \bot_{\text{sum}}(a,b) = a+b-a\cdot b$$

$$\top_{-1}(a,b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1 \\ 0, & \text{else} \end{cases} \qquad \bot_{-1}(a,b) = \begin{cases} a, & \text{if } b = 0 \\ b, & \text{if } a = 0 \\ 1, & \text{else} \end{cases}$$

The first T-norm and T-conorm are used most often, as they are simple and have some special properties (see below). The third T-norm and the corresponding T-conorm derive from probability theory.

Furthermore, the following relationships hold for any T-norm:

$$\top_{-1}(a,b) \leq \top(a,b) \leq \top_{\text{min}}(a,b)$$
$$\bot_{\text{max}}(a,b) \leq \bot(a,b) \leq \bot_{-1}(a,b).$$

In other words, every T-norm lies between the drastic T-norm (T¡sub¿-1¡/sub¿) and the minimum T-norm (T¡sub¿min¡/sub¿). Conversely, every T-conorm lies between maximum T-conorm and the drastic T-conorm.

We have discussed T-norms corresponding to AND, and T-conorms corresponding to OR. Fuzzy implication and fuzzy negation can be discussed in terms of T-norms, and will complete the connectives necessary to make statements in fuzzy logic.

Theorem 2.3.5. For any continuous t-norm, there is a unique operation $x \Rightarrow y$ such that for all $x, y, z \in [0, 1]$, we have $T(x,z) \leq y \iff z \leq (x \Rightarrow y)$. This operation is called the residuum, and is defined by$(x \Rightarrow y) = \max\{z | T(x,z) \leq y\}$.

The proof may be found in Hjek (2001)

Definition 2.3.7. A fuzzy implication is a map, $\rightarrow: [0,1] \times [0,1] \longrightarrow [0,1]$ such that

$\rightarrow$	0	1
0	1	0
1	1	1

This residuum operator is the same as the implication operator in a residuated lattice.

Definition 2.3.6. A *negation* is a nonincresing function, ν, such that $\nu(0) = 1$ and $\nu(1) = 0$. A negation can be generated by a t-norm quite simply. Given a t-norm, T, $\wedge\{y : T(x,y) = 0\}$ is a negation.

As usual, implication can be generated from an OR argument. Here, for $\perp$ a T-conorm, ν a negation, $x \rightarrow y \equiv \perp(\nu(x), y)$.

So as in classic logic, fuzzy logic uses a free propositional algebra, but uses these connectives rather than the classic connectives.

Definition 2.3.8. A *fuzzy valuation* of $P(X)$ is a proposition algebra homomorphism $v : P(X) \longrightarrow [0,1]$. Thus we may say that $p \in P(X)$ is *true with respect to* v if $v(p) = 1$ and that $p \in P(X)$ is *false with respect to* v if $v(p) = 0$.

Another interesting non-classic logic is three-valued Lukasiewicz logic. For three-valued Lukasiewicz logic, the set **F** of formulas is the same as in classical two-valued logic, however the truth evaluations are different, mapping into $0, u, 1$ instead of $0, 1$

$\neg$	
0	1
u	u
1	0

Table 2.6 NOT operation

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

Table 2.7 OR operation

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

Table 2.8 AND operation

$\rightarrow$	0	u	1
0	1	1	1
u	1	1	u
1	0	u	1

Table 2.9 implication

Fuzzy logic uses a residuated lattice for its truth set, but usually the closed interval $[0,1]$ is used in particular.

In both of these the law of excluded middle fails, and thus fuzzy logic is not a Boolean Algebra.

However fuzzy logic is a generalization of three-valued Lukasiewicz logic. In fact, we may state the following theorem about the similarity between the two.

Theorem 2.3.9. The propositional calculus for three-valued Lukasiewicz logic and the propositional calculus for fuzzy logic are the same.

Proof. For simplicity we will use $[0,1]$ for the truth degree set of fuzzy logic.
Truth evaluations are mappings A from $\mathbf{F}$ the set of formulas into the set of truth values satisfying: $A(u \wedge v) = A(u) \wedge A(v), A(u \vee v) = A(u) \vee A(v), A(v') = A(v)', \forall u, v \in \mathbf{F}$.

Two formulas are equivalent if and only if they have the same values for all truth valuations. So we need that two formulas have the same value for all truth valuations into $[0,1]$ if and only if they have the same values for all truth valuations into $\{0, u, 1\}$.

Let $\Pi := \Pi_{x \in (0,1)} 0, u, 1$ the Cartesian product with $\vee, \wedge, '$ defined componentwise. If two truth valuations from $\mathbf{F}$ into Π differ on an element, then these functions followed by the projection of Π into one of the copies of $\{0, u, 1\}$ differ on an element. Likewise if two valuations from $\mathbf{F}$ into $\{0, u, 1\}$ differ on an element, then these two functions followed by any lattice embedding of $\{0, u, 1\}$ into $[0,1]$ differ on that element. There is a lattice embedding $[0,1] \to \Pi$ given by $y \to \{y_x\}_x$ where

$$y_x = \begin{cases} 0 & \text{if } y < x \\ u & \text{if } y = x \\ 1 & \text{if } y > x \end{cases}$$

If two truth valuations from $\mathbf{F}$ into $[0,1]$ differ on an element, then these two functions followed by this embedding of $[0,1]$ into Π will differ on that element. Thus the truth values defined by the lattices $\{0, u, 1\}, [0,1]$, and Π all induce the same equivalence relation on $\mathbf{F}$, and hence yield the same propositional calculus. □

The result of this is that if you want to check whether two expressions connecting n fuzzy sets with $\vee, \wedge, '$ are equivalent in fuzzy set theory, you only need to check equality between the expressions in Lukasiewicz logic, a rather pedestrian 3^n calculations at worst.

2.3.1 Types of fuzzy sets

Starting from our original definition of a fuzzy set as a mapping $A : X \to \mathbf{L}$ with $\mathbf{L}$ a residuated lattice, $\mathbf{L} = \langle L, \wedge, \vee, 0, 1, \rangle$, we may let L be a fuzzy set. That is, we can define the membership of each item from X in our fuzzy set A as a degree to which it takes a certain value. This is what is called a *type 2* fuzzy set. A type 3 or type 4 fuzzy set may be defined by following this procedure further. A type 2 fuzzy set can be useful in mixing, for instance, probability theory with an arbitrary fuzzy set, where for some element $x \in X$, we might say that the probability to which x is an element of A is .5 with is very likely, say to a degree of .8. That is to say that if you think that there are an equal number of red and blue balls in a bag, but you are not absolutely sure, you can say define two functions to describe the situation, one associating to each ball the probability that it will be red, and another the likelihood that that probability will be correct. Up to this point when we have discussed fuzzy sets, we have meant fuzzy sets of the first type. We will continue with this convention in the rest of this thesis.

3 Applications

3.1 Thinking with fuzzy sets

Fuzzy sets and fuzzy logic is appealing when one steps back from the bivalent tenancies of mathematical ideas to the world of blurred boundaries that exists in many of the sciences and social sciences. Any definition of an object that depends on a description using linguistic hedges, their is an opening for the type of vagueness discussed in chapter 1.

In the social science, it is often the case that theories are made with vague terms. For instance, consider the statement that, ”‘A genocide or politicide is a sustained policy by states or their agents, or, in civil wars, by either of the contending authorities that results in the deaths of a substantial portion of a communal or political group.”’ Goldstone et al. (2000) The linguistic hedges of *sustained* and *substantial* require the reader to understand the sentence to mean that the duration of the policy and portion of a group in the situation are only vaguely presented, but the definition may still be applicable. Here, defining the importance of the situation in terms of a fuzzy set is appropriate. However, there have been other ways of dealing with this situation. Let us briefly discuss probability as one of these.

3.2 Differences and similarities to probability

3.2.1 Fuzzy Measures

Before we discuss the applications of fuzzy sets and fuzzy logic to thinking in social sciences, let us first briefly discuss fuzzy measure theory in order to tease apart one conception of fuzzy thinking from probabilistic thinking. Fuzzy measures were first introduced by Sugeno in 1974 (Sugeno (1974), and have since grown into a formidable aspect of fuzzy set theory. For an in depth introduction into fuzzy measure theory, see Wang and Klir (1993). First, let us remind ourselves about probability theory.

Definition 3.2.1. In probability, we consider *random* experiments to be experiments whose outcomes cannot be predicted with certainty. The *sample space* of a random experiment is a set Ω that includes all possible outcomes of the experiment; the sample space plays the role of the universal set when modeling the experiment. A *probability measure* (or *distribution*) $P : \Omega \longrightarrow [0,1]$ for a random experiment with a sample space Ω is a real-valued function defined on the collection of events that satisfies:

i. $P(X) \geq 0$ for any event X with $P(\emptyset) = 0$, and $P(\Omega) = 1$ (boundary requirements)

ii. $P\left(\bigcup_{i\in I} A_i\right) = \Sigma_{i\in I} P(A_i)$ for $\{A_i : i \in I\}$ a set of countable, pairwise disjoint events (countable additivity)

Property 1 defines the boundaries of the measure of the probability of an event to be 0 and 1, as opposed to 0 and 100, simply by convention. In general, that $P(\emptyset) = 0$ and the function be countably additive are all that is required for a function to be a *measure*. Now, an interpretation of a probability measure should be fairly familiar. Given an outcome of a random experiment denoted as event X has probability $P(X)$, one could say that, for example, if the random experiment is repeated n times, one could "'expect"' event X to be the outcome $nP(X)$ times.

In some situations, however, the randomness of probability measures is not appropriate for dealing with uncertainty. Fuzzy measures are a more generalized type of measure, which may deal with uncertainty in a different way.

Definition 3.2.2. Given a set X and a nonempty family $\mathfrak{C}$ of subsets of X, and a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1, \rangle$, a *fuzzy measure* on $\langle X, \mathfrak{C}\rangle$ is a function $g : \mathfrak{C} \longrightarrow \mathbf{L}$ such that:

i. $g(\emptyset) = 0$ and $g(X) = 1$ (boundary condition);

ii. for all $A, B \in \mathfrak{C}$, if $A \subseteq B$ then $g(A) \leq g(B)$ (monotonicity);

iii. for any increasing sequence $A_1 \subset A_2 \subset \ldots$ in $\mathfrak{C}$, if $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{C}$, then $\lim_{i\to\infty} g(A_i) = g(\bigcup_{i=1}^{\infty} A_i)$ (continuity from below);

iv. for any decreasing sequence $A_1 \supset A_2 \supset \ldots$ in $\mathfrak{C}$, if $\bigcap_{i=1}^{\infty} A_i \in \mathfrak{C}$, then $\lim_{i\to\infty} g(A_i) = g(\bigcap_{i=1}^{\infty} A_i)$ (continuity from above);

Here, as usual, out set L is often taken to be $[0,1]$.

3.2.2 possibility theory

Possibility theory traces back to Zadeh (1975) and is covered in depth by Dubois and Prade (1988), and is a different usage of possibility than that of modal logic. To understand possibility theory, we will start with a evidence theory, which is based on two dual nonadditive measures: belief measures and plausibility measures.

Definition 3.2.3. Given a universal (in our case finite) set X, a *belief measure* is a function $Bel : \mathcal{P}(X) \longrightarrow \mathbf{L}$ where $\mathcal{P}(X)$ is the power set of X and $\mathbf{L}$ is the usual residuated lattice, $\mathbf{L} = \langle L, \wedge, \vee, 0, 1, \rangle$, such that $Bel(\emptyset) = 0$ and $Bel(X) = 1$, and $Bel(\bigcup_{i=1}^{n} A_i) \geq \sum_j Bel(A_j) - \sum_{j<k} Bel(A_j \bigcap A_k) + \sum_{j<k<l} Bel(A_j \bigcap A_k \bigcap A_l) - \ldots + (-1)^{n+1} Bel(\bigcap_{i=1}^{n} A_i)$ for all possible families, $\langle A_i \rangle$ of subsets of X. Due to this inequality, belief measures are called *superadditive*. If X is infinity, the function Bel is required to be *continuous from above*.

Definition 3.2.4. Associated with each belief measure is a *plausibility measure* defined by the equation $PlA = 1 - Bel(\overline{A})$ where $\overline{A}$ is the complement of A, for all $A \in \mathcal{X}$. Thus an definition of plausibility measures independent of belief measures would be, a plausibility measure is a function $Pl : \mathcal{P}(X) \longrightarrow \mathbf{L}$ such that $Pl(\emptyset) = 0$ and $Pl(X) = 1$, and $Pl(\bigcap_{i=1}^{n} A_i) \leq \sum_j Pl(A_j) - \sum_{j<k} Pl(A_j \bigcup A_k) + \sum_{j<k<l} Pl(A_j \bigcup A_k \bigcap A_l) - \ldots + (-1)^{n+1} Pl(\bigcup_{i=1}^{n} A_i)$ for all possible families, $\langle A_i \rangle$ of subsets of X. Due to this inequality, belief measures are called *subadditive*. If X is infinity, the function Bel is required to be *continuous from below*.

Immediate consequences of super- and subadditivity are that if $n = 2$, $A_1 = A$, and $A_2 = \overline{A}$, $Bel(A) + Bel(\overline{A}) \leq 1$ and $Pl(A) + Pl(\overline{A}) \geq 1$.

Now before we discuss an interpretation of belief and plausibility measures, we must introduce a function which while not itself being a fuzzy measure, can characterize Bel and Pl.

Definition 3.2.5. A *basic probability assignment* is a function $m : \mathcal{P}(X) \longrightarrow \mathbf{L}$ such that $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$.

An interpretation of $m(A)$ for $A \in \mathcal{P}(X)$ is that $m(A)$ is the proportion to which all available and relevant evidence supports the claim that an element of X belongs to the set A in particular. Here nothing is implied for subsets $B \subseteq A$, whose basic probability assignment must be expressed by another value, $m(B)$. Notice that this is very different from probability, namely, it is not necessary that $m(X) = 1$, it is not necessary that $A \subseteq B$ imply that $m(A) \leq m(B)$, and there is no necessary relationship between $m(A)$ and $m(\bar{A})$. In fact, $m(X)$ is the proportion to which the evidence supports the claim that an element is equally likely to be in any of the subsets of X.

In order to understand belief and plausibility measures, it is important to note that a belief measure and a plausibility measure are uniquely determined by m, where for all $A \in \mathcal{P}(X)$, $Bel(A) = \sum_{B|B\subseteq A} m(B)$, and $Pl(A) = \sum_{B|A\cap B\neq\emptyset} m(B)$. The reverse direction is also possible, m may be determined from Bel (and dually from Pl) by $m(A) = \sum_{B|B\subseteq A}(-1)^{|A-B|}Bel(B)$. Thus one of the three, Bel, Pl, or m, is sufficient to determine the other two.

Now based on the relationship between m and Bel, we may say that $Bel(A)$ represents the total evidence or belief that the element belongs to A as well as subsets of A, while the $Pl(A)$ represents the total evidence or belief that an element belongs to a set which intersects in some way with A. Thus a belief measure captures the idea of belief in that our degree of belief in some statement or of some answer A to a question is equal to the sum of degrees to which we evidence supports any statement or answer which is more specific than A, and, of course, the degree to which the evidence supports exactly A. On the other hand a plausibility measure captures the idea of plausibility in that the degree of plausibility of some statement or answer A to a question is equal to the sum of the degrees to which evidence supports any set in which we are unable to determine how much more or less valid a statement independent of A is to some subset of A as well as the degrees to which evidence supports any statement or answer which is more specific than A, and, again, the degree to which the evidence supports exactly A.

With this interpretation, it is obvious that $Pl(A) \geq Bel(A)$ for all $A \in \mathcal{P}(X)$.

Definition 3.2.6. Every element $A \in \mathcal{P}(X)$ for which $m(A) > 0$ is sometimes called a *focal element of m* because the evidence supports A. Thus we may define a *body of evidence* as a pair $\langle \mathfrak{F}, m\rangle$ where $\mathfrak{F}$ is a set of focal elements and m is the associated basic probability assignment. *Total ignorance* is the case where $m(X) = 1$, and hence $m(A) = 0$ for all $A \neq X$. This also implies that $Bel(X) = 1$, and $Bel(A) = 0$ for all $A \neq X$, while $Pl(\emptyset) = 0$ and $P(A) = 1$ for all $A \neq X$.

3.3 Fuzzy Relation

We can generalize relations to fuzzy relations the same way we generalized sets to fuzzy sets. Recall, a binary relation on a set X is a subset of $X \times X$. Thus we could identify a relation, ρ, with its characteristic function, $\rho(\cdot,\cdot) : x \times X \longrightarrow \{0, 1\}$ where a pair, $(x, y) \in X \times X$ is in ρ if and only if $\rho(x, y) = 1$.

Definition 3.3.1. A *fuzzy relation*, ρ on a set X is a function $\rho : X \times X \longrightarrow \mathbf{L}$ with $\mathbf{L}$ a residuated lattice.

With this definition, a fuzzy relation, $\rho : X \times X \longrightarrow [0, 1]$, may be said to be continuous just as any function into a Euclidean space is continuous.

It may be interesting to define some properties for fuzzy relations. Let ρ be a fuzzy relation on a set X, and let $x, y, z \in X$. Then ρ is:

i. *totally reflexive* if $rho(x, x) = 1$;

ii. *totally non-reflexive* if $rho(x, x) = 0$;

iii. *symmetric* if $\rho(x, y) = \rho(y, x)$;

iv. f*-transitive* if $\rho(x, y) \precsim \rho(y, x)$ and $\rho(y, z) \precsim \rho(z, y)$ then $\rho(x, z) \precsim \rho(z, x)$;

v. $\wedge$*-transitive* if $\rho(x, y) \wedge \rho(x, z) \leq \rho(x, z)$ with $\leq$ the lattice ordering in $\mathbf{L}$;

vi. *complete* if $\rho(x, y) + \rho(y, x) = 1$.

Billot (1992) maintains that reflexivity under this definition is not necessary as in this definition as if an arbitrary does not place any importance in the comparison, they may allow $\rho \in [0, 1]$. In many cases Nguyen (1999), what we here call ”‘totally reflexive”’ is called merely ”‘reflexive”’. Nguyen (1999), Zadeh (1975). and Dubois and Prade (1980) all utilize some version of $\wedge$-transitivity, while f-transitivity is used by Billot (1992). The reason for differing definitions of transitivity is that f-transitivity maintains the irrelevance of independent alternatives.

We will concentrate on which are $\wedge$-transitive. Note, this is the case if the degree to which two elements are related is greater than the join of the degree to which each element is related to some other element. If $x \wedge y = \min x, y$ then this says that if a relation is transitive, then an item is at least as similar to another item as the minimum of the similarity between either of the items and any other item.

Definition 3.3.2. A fuzzy relation is a *fuzzy indifference relation* if it is symmetric, $\wedge$-transitive, and not totally non-reflexive.

3.3.1 fuzzy preference relation

One of the themes of this paper is to discuss each definition by the story the idea behind the definition needs to tell. If we wish to state a preference between two objects, we may not wish for that preference to be crisp. It may be important to consider the degree to which one object is preferred over another. For instance, consider the preferences, $\precsim$, between values for some country of having an exclusive trade agreement with Lichtenstein, τ_L, with Andorra, τ_A, and with the United Kingdom, τ_{UK}. Perhaps an agreement with the UK is locally preferred to an agreement with Andorra, $\precsim(\tau_{UK}, \tau_A) \geq \precsim(\tau_A, \tau_{UK})$. Perhaps $\precsim(\tau_L, \tau_A) = \precsim(\tau_A, \tau_L) = \alpha$. If $\alpha \in (0, 1]$, then we say that a country would be locally but actively indifferent of the relative values of agreements with Andorra and Lichtenstein. That is to say that the country can see that the two agreements have merit and that the merit of each is different from the other, but that the country does not particularly prefer an agreement with one to the other. On the other hand, if $\alpha = 0$, the agent cannot particularly compare the two, perhaps their is not enough information, and the decision makers of a country do not know enough about the two to know the values of such an agreement. Or perhaps the country is completely uninterested in trading with either country. Thus we will not require preference relations to be reflexive, but we will require that they not be totally non-reflexive.

Definition 3.3.3. A *fuzzy preference relation* is a fuzzy relation that is not totally non-reflexive, $\wedge$-transitive

The concept of belief functions and plausibility functions can easily be adapted to be the image of fuzzy sets, be they fuzzy preference relations, fuzzy equivalence relations, or another fuzzy mapping. Later in this thesis, I will suggest using belief functions to define fuzzy preference relations. For more on how fuzzy sets might be encoded based in any of these theories, see Klir and Yuan (1995).

4 Game Theory

In games of strategy or chance, such as chess or poker, we might talk about a strategy set as a sort of how-to book, which describes how to bet or what move to make, given the entire history of bets, moves, and cards the player has seen in the game. The goal of this book would be to guarantee each player the highest expected payoff, be it a checkmate or large winnings. If the book were complete enough, it would not be necessary to play the game, as at each stage of the game, there would be a page which dictated the subsequent move. Each player would be able to submit their strategy and instantly receive their winnings or forced to pay out their losses. In games of chance, the expected earnings and losses could also be calculated, again based on the strategy outlined in each player's book. Game theory is the tool by which we might understand and model games, foreseeing outcomes based on the rules themselves, rather than based on actually playing the game.

With this in mind, social scientists have attempted to draw parallels between games and real human situations, and have claimed that certain political or social events have happened because essentially the actors are playing some game and this outcome is parallel to some fortuitous payoff in the game. These fortuitous payoffs generally correspond to equilibria in the game, a central concept to game theory I will discuss shortly. Thus a social scientist can define some game which models the choices and preferences of the parties involved in the situation being studied, and guarantee that their theorems are logically consistent with their set of assumptions.

4.1 crisp games

Considering situations with N players, we will use strategy sets, $\Sigma = \{\Sigma_1, \ldots, \Sigma_N\}$, to describe all different strategies possible for the players. Each strategy will result in some consequence, $K = \{K_1, \ldots, K_N\}$. For each player, there is a preference relation, $\precsim = \{\precsim_1, \ldots, \precsim_N\}$, defined componentwise on the set K. Given those consequences, payoffs are doled out from a payoff set, $\Pi = \{\Pi_1, \ldots, \Pi_N\}$. A vector of consequence functions, $\vec{g} = \{g_1, \ldots, g_N\}$, then associates with each strategy a consequence, $g_i : \Sigma \longrightarrow K_i$.

Often our strategy set will be restricted to being a compact and convex subset of $\mathbb{R}^n$. Recall compact means closed and bounded. Doing this forces our strategies to be similar to a finite set in that we can make statements about its boundaries and we know that it has an upper and lower bound. For strategies of games, boundedness is sensible because although a strategy may include a dense set within a range of values, one should not expect a player to have an infinitely great amount of some resource to apply. Occasionally forcing strategy sets to be closed reduces their applicability to some game, although taking the closure of an open strategy set will often still give the researcher a clear view of the game, with the possibility of removing the points of closure after the analysis is complete.

The preference relation is defined to be a complete, transitive, and reflexive binary relation. Often, a vector of utility functions, $\vec{u} = \{u_1, \ldots, u_N\} : \Sigma \longrightarrow \Pi$ then associates a payoff with each strategy a payoff by $u_i(\sigma_i) = \pi_i$ with $\sigma_i \in \Sigma_i$ and $\pi_i \in \Pi_i$ such that for $x, y \in \Sigma_i$, $x \precsim_i y$ if an only if $u_i(x) \geq u_i(y)$.

We will discuss games as sets, $\{N, \Sigma, K, \vec{g}, \precsim\}$, letting us specify the number of players, the choices of strategies for each player, the consequences for each player given the strategies of all players, and an ordering of the preferences of each player on those consequences. Notice that preferences are defined independent of the consequences of the strategies for another player. Π_i can be any possible strategy. This means that a strategy must be defined for every possible eventuality of the game, even if that eventuality would never occur given the rest of the strategy. For instance, it will tell a chess player playing black how to play various variations of the Sicilian Defense, even if it first tells the player to respond to an opening of e4 with e5, preferring the Ruy Lopez opening.

Often situations are modeled by games to show that some set of strategies by the players is more likely than others. One reason that a strategy would be more likely is that given some a strategy for all other players, no one player would secure for themself a greater payoff by changing their strategy.

Definition 4.1.1. for some N-player game, $\{N, \Sigma, K, \vec{g}, \precsim\}$, with Σ the set of strategy profiles, Π set of payoff profiles, $\vec{u} : \Sigma \longrightarrow K$ the utility function, a strategy profile, $\sigma^* = \{\sigma_1^*, \ldots, \sigma_N^*\}$ is a *Nash equilibrium* if for all $i \in \{1, \ldots, N\}$, there exists no strategy $\Sigma_i \neq \Sigma_i^*$ such that $g_i(\{\Sigma_1^*, \ldots, \Sigma_{i-1}^*, \Sigma_i, \Sigma_{i+1}^*, \ldots, \Sigma_N^*\}) \precsim_i g_i(\{\Sigma_1^*, \ldots, \Sigma_N^*\})$

This is the basic equilibrium condition, and has been the most explored equilibrium in Game Theory. Another way of formulating a Nash equilibrium is by defining a best-response function. A best-response function is a function that takes as its input the strategy of all of the opponents of some player and outputs the one or more strategies which will give the player the highest payoff.

Definition 4.1.2. In a N-player game, given a strategy set $\Sigma = \{\Sigma_1, \ldots, \Sigma_N\}$, and a consequence function for player i, g_i a *best-response function* is a function $B_i : \{\Sigma_1, \ldots, \Sigma_{i-1}, \Sigma_{i+1}, \ldots, \Sigma_N\} \longrightarrow \Sigma_i$ such that $B_i = \{\sigma_i \in \Sigma_i : g_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i \sigma_{i+1}, \ldots, \sigma_N) \precsim g_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i^*, \sigma_{i+1}, \ldots, \sigma_N)$ for all$\sigma_i^* \in \Sigma_i\}$

We can define Nash equilibria in terms of best-response functions.

Lemma 4.1.3. for some N-player game, $\{N, \Sigma, K, \vec{g}, \precsim\}$, with Σ the set of strategy profiles, K set of consequences, $\vec{g} : \Sigma \longrightarrow K$ the vector of consequence functions, and $\precsim = (\precsim_1, \ldots, \precsim_N)$ the set of preference relations for each player, a strategy profile, $\sigma^* = \{\sigma_1^*, \ldots, \sigma_N^*\}$ is a *Nash equilibrium* if and only if for all best-response functions, $\{B - 1, \ldots, B_N\}$, $\sigma_i \in B_i((\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ for all $i \in \{1, \ldots, N\}$

To explore these concepts further, we will need some concepts from topology.

Definition 4.1.4. A preference relation, $\precsim_i$, over a set, X, with $|X| = n$, is *quasiconcave on* X_i if for every $x^* = \{x_1^*, \ldots, x_n^*\} \in X$, the set, $\{x_i in X_i : (x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*) \precsim (x_1^*, \ldots, x_n^*)\}$ is convex.

One of the most famous results in Game theory is John Nash's 1951 proof of the existence of Nash equilibria in N-player games given certain conditions. His proof used the Kakutani fixed point theorem, which gives conditions on a function such as the best-response function so that there exists some value $\sigma^* = \{\sigma_1^*, \ldots, \sigma_N^*\}$ such that $\sigma_i^* \in B_i((\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_N^*)$ for all $i \in \{1, \ldots, N\}$.

Theorem 4.1.5. Kakutani's fixed point theorem Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : x \longrightarrow X$ be a set-valued function such that for all $x \in X$, the set $f(x)$ is nonempty and convex and the graph of f is closed (for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \longrightarrow x$, and $y_n \longrightarrow y$, we have $y \in f(x)$, then there exists a fixed point $x^* \in X$ such that $x^* \in f(x^*)$.

Using fixed point theorems gives a situation that is invariant over repeated applications of a mapping. In this case, we will find that we may continue to apply the best-response function, but we will reach a particular state which no player has a profitable deviation as defined by the response function. Now we can state and prove a theorem about the existence of Nash equilibria.

Theorem 4.1.6. For some N-player game, $\{N, \Sigma, K, \vec{g}, \precsim\}$, a Nash equilibrium exists if for all $i \in \{1, \ldots, N\}$ the set Σ_i of strategies for player i is a nonempty compact, convex subset of $\mathbb{R}^n$, and each preference relation, $\precsim_i$ is continuous and quasiconcave.

Proof. Define $B : \Sigma \longrightarrow \Sigma$ by $B(\sigma) = \Pi_i B_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ for all $i \in \{1, \ldots, N\}$ where B_i is the best response function for player i. Notice $B_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ is nonempty since $\precsim_i$ is continuous and Σ_i is compact. Also, $B_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ is convex by the definition of the quasiconcavity of $\precsim_i$ on Σ_i. Also, B has a closed graph by the closedness of $\vec{g}$. Notice, this is equivalent to saying that for sequences of strategies, $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$, where $P_n \longrightarrow P$ and $Q_n \longrightarrow Q$ that if for all n, $Q_n \in B(P_n)$, then $Q \in P$. Thus, Kakutani's fixed point theorem applies and B has a fixed point. So by our lemma, this fixed point is a Nash equilibrium of the game.

In the case of two player games, a certain number of examples are illustrative of the properties required for a game to have a Nash equilibrium. The games in the next sections are given in the form of a matrix, with the choices for player 1 at the top and for player 2 along the side, and the payoffs for a given pair of choices given as the corresponding ordered pair such that (x,y) implies that player 1 gets a payoff of x, and player 2 gets a payoff of y. Each players preferences are here to get a higher numbered payoff.

4.1.1 Bach or Stravinsky (Battle of the Sexes)

The game classically known as the Battle of the Sexes or more recently dubbed Bach or Stravinsky involves two players who would like to coordinate their actions so that both end up at the same venue for a rendez-vous. It is a case where a game has two Nash-Equilibria, since given the choice of one player, the other will want to chose the same composer, that is the best response function will recommend the same composer as the other player.

	Bach	Stravinsky
Bach	(5,6)	(0,0)
Stravinskyy	(0,0)	(6,5)

Table 4.1 Bach or Stravinsky

4.1.2 Matching Pennies

Matching pennies is the classic zero-sum game, which means that the two players have antisymmetric payoffs. In this game, both players have a penny, which they conceal from their opponent. The they simultaneously show their opponent their penny, having placed it heads up or tails up. Player 1 gets a point if the pennies match, while player 2 receives a point if the pennies do not match. This example shows a case where their is no Nash equilibrium. In this case, this is because the preference relation is not quasiconcave.

	Heads	Tails
Heads	(1,-1)	(-1,1)
Tails	(-1,1)	(1,-1)

Table 4.2 Matching Pennies

4.1.3 Prisoner's Dilemma

The most famous game is perhaps the Prisoner's Dilemma. In contemporary literature, it first arose in work at RAND Corporation such as that of Flood (1952). Its application was popularized largely by Schelling (1960). It has also been very popular in a repeated form (Axelrod (1985)) where one can study concepts like learning and reputation. Borges et al. (1997) used fuzzy rules to study this repeated or iterated prisoner's dilemma. Many stories have been told to describe situations where this arises. Perhaps the most interesting (if not compelling) is in the situation of an arms race. In this telling, one would hope that the two players (countries) cooperate and halt production or dismantle their arms. However if one country does so, the best response function tells the other to get themselves an advantage by defecting (and building a large enough arsenal to destroy their opponent or at least to force the opponent into a weak bargaining position). Thus if neither player trusts the other, they will both defect, the unique Nash equilibrium, even though that position is not efficient (Pareto optimal).

	Cooperate	Defect
Cooperate	(5,5)	(8,1)
Defect	(1,8)	(3,3)

Table 4.3 Matching Pennies

4.2 fuzzy games

We saw before that sets can be turned into fuzzy sets and relations can be turned into fuzzy relations. Thus, it seems to create a fuzzy game, we have many choices. Each choice stems from a different story about how the game is to be played. Song and Kandel (1999) examines a variation on the prisoner's dilemma where the degree to which the player wishes to help or harm his partner is fuzzy. Garagic and Cruz (2003) uses a fuzzy fixed-point theorem to show that certain fuzzy matrix games have a Nash equilibrium. Another way of making a game fuzzy is explored by Arfi (2006) where outcomes of a variation of the prisoner's dilemma are broken down into finite, discrete values based on finite, discrete levels of cooperation and defection. Another work, Nishizaki and Sakawa (2001) explores cooperative fuzzy games as a method of conflict resolution and concentrates on numerical solutions to such game. In our case, we will start with looking at a simple fuzzy game by utilizing a fuzzy preference relation. In this case, we might say that the degree to which a player prefers a given object over another is fuzzy. One way to do this, is to relate each outcome by the degree of belief that it is better than another outcome. In this case, we need to define a new type of best-response function.

Definition 4.2.1. In a N-player game with fuzzy preference relation, $\precsim$, given a strategy set $\Sigma = \{\Sigma_1, \ldots, \Sigma_N\}$, and a consequence function for player i, g_i a *fuzzy preference relation best-response function* is a function $B_i : \{\Sigma_1, \ldots, \Sigma_N\} \longrightarrow \Sigma_i$ such that $B_i(\sigma_1, \ldots, \sigma_N) = \{\sigma_i^* \in \Sigma_i : (\sigma_1, \ldots, \sigma_N) \precsim (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i*, \sigma_{i+1}, \ldots, \sigma_N)\}$

The fuzzy preference relation best-response function is obviously non-empty. In this case, our previous proof of the existence of Nash equilibria can still hold so long as certain conditions hold:

Definition 4.2.2. A fuzzy preference relation, $\precsim_i$, over a set, X, with $|X| = n$, is *fuzzy quasiconcave on* X_i if for every $x^* = \{x_1^*, \ldots, x_n^*\} \in X$, the set, $\{x_i in X_i :\precsim ((x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*), (x_1^*, \ldots, x_n^*)) \geq \precsim ((x_1^*, \ldots, x_n^*), (x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*))\}$ is convex.

Theorem 4.2.3. For a fuzzy preference relation, $\precsim$, on an N-player game, $\{N, \Sigma, K, \vec{g}, \precsim\}$, a Nash equilibrium exists if for all $i \in \{1, \ldots, N\}$ the set Σ_i of strategies for player i is a nonempty compact, convex subset of $\mathbb{R}^n$, and for each preference relation, $\precsim_i$,

i. $\precsim$ is continuous;

ii. $\precsim$ is fuzzy quasiconcave.

The proof follows exactly as before.

Clearly, this conception of fuzzy games amounts to reducing fuzzy preferences to crisp preferences. Another very similar construction called an equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal is discussed in Nishizaki and Sakawa (2001). There work concentrates on computational methods for finding such solutions. We turn to a more sophisticated conception of a fuzzy game, and define a Nash equilibrium concept based upon it. A generalized version of this is due to Butnariu (1979). Exploring this work will give us another example of the importance of fixed points in analyzing equilibria in game theory. As Butnariu's formulation is more general than that of some other games found in more recent research, including that of Nishizaki and Sakawa, examples and ideas discussed in that work could be adapted to Butnariu's concept. However, we will mention an example from Butnariu's work which shows that fuzzy beliefs about a players opponents can allow us to explore interesting games that allowing for only fuzzy preference relations among outcomes misses.

4.2.1 Butnariu's Game

To start, we will explore the concept of fuzzy topology and of a fuzzy fixed point theorem.

Definition 4.2.4. A *fuzzy correspondence* on a set X is a mapping $R : X \longrightarrow \mathcal{L}(X)$ where $\mathcal{L}(X)$ is the fuzzy set based on the powerset of X. We may examine the membership function, $a(\cdot)$, of the fuzzy subset $A \in \mathcal{L}(X)$. For a fuzzy correspondence, $R(\cdot)$, on X, we use R_X to denote the fuzzy subset of $X \times X$ which has a membership function $r(x, y)$ for any $(x, y) \in X \times X$. We note $M(X)$ the set of fuzzy correspondences on X.

Definition 4.2.5. Now, given a fuzzy correspondence $R(\cdot)$ on X, a fixed point, $x^* \in X$, is an element such that for all $x \in X$, $r(x^*, x^*) \geq r(x^*, x)$.

Definition 4.2.6. We say that a fuzzy subset, $A \in \mathcal{L}(X)$, is *convex* if and only if its membership function is concave, ie for all $q \in [0, 1]$ and for all $(x, y) \in X \times X$, we have $a(qx + (1 - q)y) \geq \min a(x), a(y)$.

Remember that a *topological vector space*, X, is a vector space with continuous vector addition and scalar multiplication. Such a topological space is *Hausdorff-separated* if for any $x \in X$, the intersection of all the closed neighborhoods of X is the single element $\{x\}$. Such a space is *locally convex* if for all $x \in X$, every neighborhood of x is convex.

Definition 4.2.7. A fuzzy correspondence $R(\cdot)$ is *convex* if for the topological vector space X, locally Hausdorff-separated and $C \subseteq X$ such that C is nonempty, compact and convex, the fuzzy subset $R_x \in \mathcal{L}(X)$ is a convex fuzzy subset of X. $R(\cdot)$ is closed if and only if the membership function, $r(x, \cdot)$ is upper semicontinuous.

Now, following the lead of Butnariu (1979) and Heilpern (1981), we present a fixed point theorem in a fuzzy universe:

Theorem 4.2.8. Butnariu's generalized fuzzy fixed point theorem For X a real locally convex, Housdorff-separated, topological vector space, X, and any nonempty, compact, convex subset, $C \in X$, if $R(\cdot)$ is a convex fuzzy correspondence closed in C, then $R(\cdot)$ has a fixed point in C.

The proof can be found in Billot (1992), and is an adaption of the one given in Butnariu (1979) and Butnariu (1978). It is different from the theorem and proof given by Heilpern (1981), which requires that the fuzzy correspondence be, in effect, a fuzzy contraction mapping on an appropriately defined fuzzy metric. Notice that Butnariu's theorem is valid for any topological structure, including metric spaces which are used in the fuzzy fixed point theorem of Song and Kandel (1999). If we restrict X to a subset of a Euclidean space and the fuzzy correspondence is a nonempty point to set function, we have Kakutani's fixed point theorem.

We also need to impose some particular characteristics on our fuzzy discourse. We will denote U as the universe of objects being discussed. as before, for $x \in U$, $a(x)$ is the membership level of x. The product of two fuzzy subsets, A and B with membership level functions, $a(\cdot)$ and $b(\cdot)$ respectively, is defined by $[a \times b](x) = a(y) \times b(z)$ for $x = (y, z) \in U$ and $[a \times b](x) = 0$ else. We will define a particular product of membership levels of fuzzy subsets where for fuzzy relations, $R : A \longrightarrow B$ and $S : B \longrightarrow C$, and $(x, y) \in A$, we have $S \circ R(x, z) = \sup R(x, y) \times S(y, z) : y \in U$ for $(x, z) \in U$, $S \circ R(x, z) = 0$ else.

Now, again, we return to the stories that are told to explain why grouping of mathematical structures models some sort of game. Again we will consider N-person games. In this game, all players first gather and exchange information about what they will do, based on whatever logic they wish to employ. Then all the players are cut off from communication with each other to some degree (possibly completely).

At this point, each player knows the choices of strategies that other players have. They can then define for themselves a vector of fuzzy sets which express the degree to which they believe the other players might choose some strategy. The players will also be unsure about whether any given outcome is preferable to another.

i. $\Sigma = \{\Sigma_1, \ldots, \Sigma_N\}$ will be the set of strategies. We will require each player, i, to have only a finite number of strategies from which to choose from, indexed by the function, $n(i)$. so Σ_i is a vector of strategies with $n(i)$ elements.

ii. For each player, i, we will define the *strategic arrangement* as a set $Y_i \in [0,1]^{n(i)}$ such that $Y_i = \Pi_{j=1}^{n(i)} Y_i^j$, $\Sigma_{j=1}^{n(i)} Y_i^j = 1$, where Y_i^j is the percent of time player i chooses action j. Let $Y = \Pi_{i=1}^{N} Y_i$. This allows the players to choose mixed strategies, and allows the set of strategies to be convex.

iii. For each player, i, we define $e_i \in \mathcal{L}(\Pi_{i\in N} Y_i)$ and for all $w = (w_1, \ldots, w_N) \in \Pi_{i\in N} Y_i$, as a *belief level* of the strategic choice w evaluated by player i. That is to say that each player evaluates the degree to which they believe that the other players will play each of their possible strategies. e_i is a belief function as defined earlier.

iv. We will be considering $s_i = (A_i, w_i) \in \mathcal{L}((\Pi_{j\in N-i} Y_j) \times Y_i)$ as a *strategic conception* for player i. Here, A_i is the belief of player i in the strategic arrangements of the other players, and is a fuzzy subset of the set of all of the possible strategies of the other players. Here A_i represents player i's belief in the strategies that would be chosen by the other players given that i plays w_i.

v. We also want it to be the case that for $A_i \in \mathcal{L}(\Pi_{j\in N-i} Y_j)$ and $A_i \neq \emptyset$ then there exists $(A_i, w_i) \in \mathcal{L}(\Pi_{j\in N-i} Y_j) \times Y_i)$ such that $e_i[A_i](w_i) \neq 0$. Thus no matter what the others might do, no player will be unable to formulate a strategy.

Definition 4.2.9. A *play* in this game is a vector $\vec{s} = (s_1, \ldots, s_N)$ such that $s_i \in \mathcal{L}(\Pi_{j\in N-i} Y_j) \times Y_i$. For $s_i = (A_i, w_i) \in \mathcal{L}(\Pi_{j\in N-i} Y_j) \times Y_i)$ and $s_i^* = (A_i^*, w_i^*) \in \mathcal{L}(\Pi_{j\in N-i} Y_j) \times Y_i)$, s_i is a *better strategic conception* than s_i^* if and only if $e_i[A_i](w_i) > e_i[A_i^*](w_i^*)$. s is *socially preferred* to s^* if s_i is a better strategic conception than s_i^* for all $i \in N$.

Definition 4.2.10. A *possible solution* of a game of this type is a play, $s^* = (s_1^*, \ldots, s_N^*)$, where $s_i^* = (A_i^*, w_i^*)$ such that for any other play, $s = (s_1, \ldots, s_N)$, with $s_i = (A_i^*, w_i)$, $e_i[A_i^*](w_i^*) \geq e_i[A_i^*](w_i)$.

Thus a possible solution corresponds intuitively with equilibrium points of non-fuzzy games in that it is marked by its advantage in terms of a preference relation. In games of this type, an equilibrium point is a refinement of this concept.

Definition 4.2.11. An *Nash equilibrium* of a game of this type is a possible solution, $s^* = (s_1^*, \ldots, s_N^*)$, where $s_i^* = (A_i^*, w_i^*)$ such that $a_i^*(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N) = 1$ if $w_i = w_i^*$ for any $i \in N - i$ and $a_i^*(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N) = 0$ elsewhere.

The additional condition, often called *non-constraining collaboration* implies that player i expect a strategy w_j from player j only when player j intends on playing that strategy. Thus a Nash equilibrium arises after uncertainty in the belief of the moves of the other players is removed.

We will now compute two new fuzzy preference relations:

Definition 4.2.12. The *individual fuzzy preference* of a player i is the fuzzy relation $E_i(\cdot, \cdot)$ on $\mathcal{L}(W \times W)$ where $W = \Pi_{j \in N} Y_j \times Y$ such that $E_i(s, s^*) = [e_i[A_i](w_i^*) \wedge e_i[A_i](w_i)] \times \Pi_{j \in N - \{i\}} e_j[A_j](w_j^*)$.

This individual fuzzy preference now takes into account what the other players would believe about the degree of possibility of player i going against their expectations.

Definition 4.2.13. The *social preference relation* is a fuzzy relation $E(\cdot, \cdot)$ on $\mathcal{L}(W \times W)$ such that $E_s(s, s^*) = \Pi_{i \in N} E_i(s, s^*)$ for all $(s, s^*) \in W \times W$.

Now we may state and prove the main theorems presented originally by (**?**, Butnariu79)

Theorem 4.2.14. Let s^* be a play of a game of this type, with $s_i^* = (A_i^*, w_i^*)$. If $A_i^* \neq \emptyset$, for all i, then s^* is a possible solution if and only if s^* is a fixed point in $E_s(\cdot, \cdot)$.

Lemma 4.2.15. There is no i such that if s^* is a fixed point in $E_s(\cdot, \cdot)$, then for $s_i^* = (A_i^*, w_i^*)$, we have $e_i[A_i^*](w_i^*) = 0$.

It leaves to be understood whether any useful equilibrium concepts can be defined and shown to exist if the strategic choices are fuzzy. We remember that a Nash equilibrium arises when for all i, $j \in N$, $A_i^* = A_j^*$. We would call this game a game with perfect information. However, the preference relations remain possibly fuzzy, and we have reduced out fuzzy game to the fuzzy preference relation game discussed earlier. This suggests that the players may approach a Nash equilibrium in a fuzzy game of this type as they exchange more information about their full range of strategic conceptions.

4.2.2 Example

Let us look at an example of an application of his formulation and equilibrium concept in political science based on that provided by Butnariu (1979). For simplification, we can look at two players named 1 and 2. In this game, the two players are interested in defining their military investments vis-a-vis each other. Here we are looking at only the international level game in Putnam's two level game concept, ignoring completely the influence of national bodies such as special interests and legislatures. For the purposes of this game, we will say that military investments perceived by opponents serve as a deterrence for escalating of military conflict between the states. The two states then have as goals both the reduction of intensity of conflict and the minimization of economic energy put into military investments toward that ends.

We can easily arrange this situation so that it fits in the conditions defining Butnariu's game. First, let $\Sigma_k = \Sigma_1^k, \ldots, \Sigma_{n(k)}^k$ be different strategies of military investment for player k. Perhaps Σ_1^k is missile defense, Σ_2^k is increased naval size, Σ_3^k is greater nuclear capabilities, and so on. Note that this set is crisp, and the players are allowed to chose as strategic arrangements mixed strategies which amount to investing a certain amount of a budget in any of various strategies until the budget is used up. In the course of relations between the two players, each players can exchange information and (possibly) make guarantees about their future actions. From these exchanges, each player can decide what they believe to be the value of their actions basing these values on their beliefs about how their opponent values the actions available to them. Normalizing each players budget, we can denote a strategic arrangement by $Y_k \in [0, 1]^{n(k)}$. We notice that the set of degrees of belief of player k in the plausibility of each of their opponents strategies Σ^{-k} defines a fuzzy set. We may denote these belief by $e_i \in \mathcal{L}(\Pi_{i \in N} Y_i)$. Fuzzy pairs, $(A_k, w_k) \in (Y_{-k} \times Y_k)$ are strategic conceptions, where given player k chooses w_k, A_{-k} is a fuzzy set representing player k's level of belief that his opponent will play any given strategy given k chooses w_k. We will also let it be the case that for any set of information exchanged, each player can form a strategic conception as a reply to his beliefs about the other players actions, ie that the fifth condition is satisfied.

This example, then, clearly defines a fuzzy game of the type Butnariu described. Thus we can say that two strategic conceptions of the game can be made, and one can be preferable to the other if its belief level is more believable. Each player will examine to what degree they believe their opponent will do something advantageous given they act in a way that most closely approaches what they believe is advantageous to them. We will again be able to define a possible solution as before, a condition where both players find their play to be advantageous compared with any other conception. It is not clear exactly what these possible solutions will look like in terms of the military investments of the players without making explicit the belief functions and strategies available, but our theory shows that the possible solutions do arise as fixed points of fuzzy correspondences. So if given a chance to play multiple games, we may expect the players to eventually discover a stable solution by iteration, a stable play, which represents a possible solution. This is by no means a necessary outcome, as the eistence of a fixed point does not imply that it will arise in a finite number of iterations of our fuzzy corespondence. Butnarius definition of an equilibrium point is also meaningful here, as it is a possible solution where the uncertainty of each players beliefs in the actions of the other players is removed. In this case, it means that it is a possible solution where the military apparatus of the opponent is openly observed, perhaps by cameras or UN appointed monitors. Butnariu's work, then, suggests that it is these solutions that are the most stable. It is not, of course, surprising that reducing uncertainty provides stability. Unfortunately, most of the work in fuzzy game theory has not brought us closer to examining games with fuzzy beliefs, only fuzzy goals. A return to Butnariu's work, then, may allow us to reexamine what it is about fuzzyness that makes it so attractive to game theory.

5 Conclusion

In this paper we have defined fuzzy sets an looked at a couple ways they have been used, eventually discussing the topic of fuzzy games. We concentrated especially on Butnariu's game because it encompasses the more detailed work of other authors, and also because the example Butnariu's game provides is particularly seductive to those interested in social science. In particular, Butnariu examines fuzzyness in social beings (people, societies, etc.) beliefs about the actions of social beings, including themselves. It is true that Butnariu's game suggests that to a certain degree, more information brings more stability to games, a fact that is well known both in the study of theoretical games of a crisp type (although not universally), as well as intuitively and in social practice (again exceptions do exist). An exploration of social events with an eye to their formulation in the terms of Butnariu's game has not been seen, although parallel formulations do exist, and suggest that Butnariu's theorems would hold true in the real world. In this paper, I did not discuss the nature of the fuzzy correspondences which would give rise to possible solutions of Butnariu's game, and in future work, both a discussion of that nature, and an exploration of what would make that correspondence easily definable in terms of real world processes would be important. This would allow us to compare the mechanisms which make Butnariu's theorems work mathematically with the mechanisms which govern social processes in the real world. This type of comparison, along with an honest assessment of how all the theorems of game theory speak to actual causal mechanisms in the real world applications social science is so keen on, is perhaps the next important movement in the philosophy of social science. I personally hope to see more work on the meanings of the applications of game theory to social science. Why does game theory model human behaivior? What mathematical tools can we use to help us better make those models? Butnariu's game shows that game theory is still relevant if we release our formulations from the constraints of crisp sets, and where fuzzy sets better model a situation than crisp sets, the existance of a maturing field of fuzzy game theory will give social scientists the tools to more often move in that direction.

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