Cooperative Game Theory: Applications

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Für Corinna, Ben, Jasper, Samuel

Preface

What is this book about?

This book is on applied cooperative game theory. We deal with agents exchanging objects, profit centers within firms, political parties groping for power and many other sorts of "players". Subsets of players are usually called coalitions.

Cooperative game theory focuses on the question of "who gets how much". This question is determined by the two pillars of cooperative game theory. The first pillar is the coalition function (also called characteristic function) that describes the economic (sociologic, political) opportunities open to all the possible coalitions. Thus, a coalition function may represent a bargaining situation, a market, an election, a cost-division problem and so on.

The second pillar is the solution concept applied to coalition functions. Solutions consist of payoffs attributed to the players. Typically, solutions can be described in one of two ways. Either we provide a formula or an algorithm that tells us how to transfer a coalition function into payoff vectors (formula definition). Or we put down axioms that describe in general terms how much players should get (axiom definition) – axioms in cooperative game theory are general rules of division. For example, Pareto efficiency demands that the worth of the grand coalition (all players taken together) is to be distributed among the players. According to the axiom of symmetry, symmetric (not distinguishable but by name) players should obtain the same payoff.

Ideally, the formula and the axiom definitions coincide. This means that a solution concept can be expressed by a formula or by a set of axiom and that both ways are equivalent – they lead to the very same payoff vectors.

While we also talk about matching formula and axiom definitions, we stress applications over theory. This means that we deal with theoretical concepts only if they are helpful for the applications that we have in mind. The knowledgeable reader will excuse us for omitting the von Neumann-Morgenstern sets or the nucleolus. Instead, the Shapley value and derivatives of the Shapley value take center stage.

Which applications do we cover?

We deal with many different institutions that range from markets and elections to coalition governments and hierarchies. In particular, we consider the following applications.

- How does the price obtained on markets depend on the relative scarcity of the traded objects?
- How can we model power and power-over?
- Can we expect unions to be detrimental to employment?
- Will unemployment benefits increase unemployment?
- How can overhead costs be shared?
- How does the number of ministries a party within a government coalition obtains depend on hte number of seats in parliament?

What about mathematics ... ?

Cooperative game theory is not too demanding in terms of mathematical sophistication. Also, since we have an applied focus, we are more interested in interpretation and application than in axiomatization.

Exercises and solutions

The main text is interspersed with questions and problems wherever they arise. Solutions or hints are given at the end of each chapter. On top, we add a few exercises without solutions.

Thank you!!

I am happy to thank many people who helped me with this book. Several generations of students were treated to (i.e., suffered through) continuously improved versions of this book. Frank Hüttner and Andreas Tutic ... I also thank my coauthors Andre Casajus, Tobias Hiller and ... for the good cooperation with high payoffs to everyone. Some generations of Bachelor and Master students also provided feedback that helped to improve the manuscript.

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Harald Wiese

Introduction and Pareto efficiency

Part A

Introduction and Pareto efficiency

CHAPTER I

Introduction

1. The players, the coalitions, and the coalition functions

Throughout the book, we deal with a player set $N = \{1, ..., n\}$ and the subsets of N which are also called coalitions. Thus, the coalitions of $N := \{1, 2, 3\}$ include $\{1, 2\}$, $\{2\}$, $\emptyset$ (the empty set – no players at all) and N (all players taken together – the grand coalition).

The general idea of cooperative game theory is that

- coalition functions describe the economic, social or political situation of the agents while
- solution concepts determine the payoffs for all the players from N taking a coalition function as input.

Thus,

	coalition functions
+	solution concepts
yield	payoffs.

In the literature, there are two different sorts of coalition functions, with transferable utility and without transferable utility. We focus on the simpler case of transferable utility in all parts of the book except the last one. In the framework of transferable utility, a coalition function v attributes a real number $v(K)$ to every coalition $K \subseteq N$. Consider, for example, the gloves game v for $N = \{1, 2, 3\}$ where the two players 1 and 2 hold a left glove and player 3 holds a right glove. The idea behind this game is complementarity – pairs of gloves have a worth of 1. Thus, the coalition function for that gloves game is given by

$$\begin{aligned} v(\emptyset) &= 0, \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0, \\ v(\{1,2\}) &= 0, \\ v(\{1,3\}) &= v(\{2,3\}) = 1, \\ v(\{1,2,3\}) &= 1. \end{aligned}$$

Left-glove holders and right-glove holders can stand for the two sides of a market – demand and supply. For example, the left-glove holders buy right gloves.

2. The Shapley value

In our mind, the Shapley value is the most useful solution concept in cooperative game theory. First of all, it can be applied directly to problems ranging from bargaining over cost division to power indices. Applying the Shapley value to the above gloves game yields the payoffs

$$Sh_1(v) = \frac{1}{6}, Sh_2(v) = \frac{1}{6}, Sh_3(v) = \frac{2}{3}.$$

We see that the Shapley value

- distributes the worth of the grand coalition $v(N) = 1$ among the three players ($Sh_1(v) + Sh_2(v) + Sh_3(v) = 1$),
- allots the same payoff to players 1 and 2 because they are "symmetric" ($Sh_1(v) = Sh_2(v)$), and
- awards the lion's share to player 3 who possesses the scarce resource of a right glove.

Thus, the Shapley value tells us how market power is reflected by payoffs. This and many other applications are dealt with in the first part of our book which concentrates on the Shapley value (and some related concepts such as the Banzhaf value).

There are several alternative ways to calculate the Shapley value. Let us denote the payoff to player i by x_i. Assume the players 1, 2 and 3 bargain on how to divide the worth of the grand coalition, $v(N) = 1$, between them, i.e., we have

$$x_1 + x_2 + x_3 = 1. \quad \text{(I.1)}$$

Furthermore, let every player use a "where would you be without me" argument. In particular, player 3 could issue the following threat to player 1 (and similarly to player 2): "Without me, there would be only two left gloves and your payoff would be zero rather than x_1, i.e., you, player 1, would lose

$$x_1 - 0$$

without me."

Player 1's counter-threat against player 3 runs as follows: "Without me, you, player 3 would find yourself in an essentially symmetrical situation with player 2 (one right-hand glove versus one left-hand glove) and obtain the payoff $\frac{1}{2}$, i.e., you would lose

$$x_3 - \frac{1}{2}$$

without me."

The Shapley value rests on the premise of equal bargaining power – both arguments carry the same weight. Thus, the two differences are the same:

$$\underbrace{x_1 - 0}_{\substack{\text{loss to player 1} \\ \text{if player 3 withdraws}}} = \underbrace{x_3 - \frac{1}{2}}_{\substack{\text{loss to player 3} \\ \text{if player 1 withdraws}}} . \tag{I.2}$$

Since we have analogous threat and counter-threat between players 2 (rather than player 1) and 3, we find

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 \text{ (eq. I.1)} \\ &= \left(x_3 - \frac{1}{2}\right) + \left(x_3 - \frac{1}{2}\right) + x_3 \text{ (eq. I.2)} \end{aligned}$$

and hence

$$(x_1, x_2, x_3) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right). \tag{I.3}$$

The Shapley value is easy to handle. This simplicity gives room for additional structure that may be needed in applications. Thus,

- different players may belong to different groups that work together, bargain as a group etc.
- any two players may or may not be linked together where the links stand for communication or cooperation.

We will briefly introduce these

Shapley + structure

in this introductory chapter and treat them in some detail in later chapters.

3. The outside option value

Taking up the gloves game again, assume that the glove traders 1 (left glove) and 3 (right glove) agree to cooperate to form a pair of gloves. We can express this fact by the partition of N

$$\{\{1,3\},\{2\}\}$$

where we address $\{1,3\}$ and $\{2\}$ as that partition's components.

What are the player's payoffs in such a situation? The first idea might be to apply the Shapley value to the individual components. In fact, the resulting value is known as the AD value (where A stands for Aumann and D for Dreze) and given by

$$AD_1(v) = AD_3(v) = \frac{1}{2}, AD_2(v) = 0.$$

More recent developments in cooperative game theory point to the fact that player 3 should obtain more than $\frac{1}{2}$ because he can threaten to join forces with player 2 rather than player 1. Thus, player 2 is an "outside option" for player 3.

How can we find the player's payoffs in that case? First of all, players 1 and 3 will share the value of a glove, i.e., we have

$$x_1 + x_3 = 1 \tag{I.4}$$

and $x_2 = 0$. When 1 and 3 bargain on how to share the payoff of 1, both players may point out that each of them is necessary to form the component $\{1,3\}$. Therefore, the gain from leaving player 2 out should be divided equally where the Shapley value (for the trivial partition $\{\{1,2,3\}\}$ serves as a reference point:

$$\underbrace{x_1 - Sh_1(v)}_{\substack{\text{gain for player 1} \\ \text{from forming component } \{1,3\}}} = \underbrace{x_3 - Sh_3(v)}_{\substack{\text{gain for player 3} \\ \text{from forming component } \{1,3\}}}. \tag{I.5}$$

By

$$\begin{aligned} 1 &= x_1 + x_3 \text{ (eq. I.4)} \\ &= [x_3 - Sh_3(v) + Sh_1(v)] + x_3 \text{ (eq. I.5)} \\ &= 2x_3 - \frac{2}{3} + \frac{1}{6} \text{ (eq. I.3)} \end{aligned}$$

we obtain the outside-option value payoffs due to Casajus (2009)

$$(x_1, x_2, x_3) = \left(\frac{3}{4}, 0, \frac{1}{4}\right).$$

4. The network value

Instead of considering partitions, we may assume a network of links between players. A link between two players means that these two players can communicate or cooperate. The corresponding generalization of the Shapley value is known as the Myerson value.

Departing fromt the gloves game, we assume that players 1 and 3 are the productive or powerful players. This is reflected by the coalition function v given by

$$v(K) = \begin{cases} 1, & K \supseteq \{1,3\} \\ 0, & \text{otherwise} \end{cases}$$

Coalitions different from $\{1,3\}$ and $\{1,2,3\}$ have the value zero. Without the network, we should expect the Shapley payoffs $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$:

- Player 2 is unimportant (a null player, as we will say later) and obtains nothing.
- The two players 1 and 3 are symmetric and share the worth of 1.

However, we assume restrictions in cooperation or communication, In particular, players 1 and 3 are not directly linked (see the upper part of fig. 1). Player 2's role is to link up the productive players 1 and 3. How should he be rewarded for his linking service?

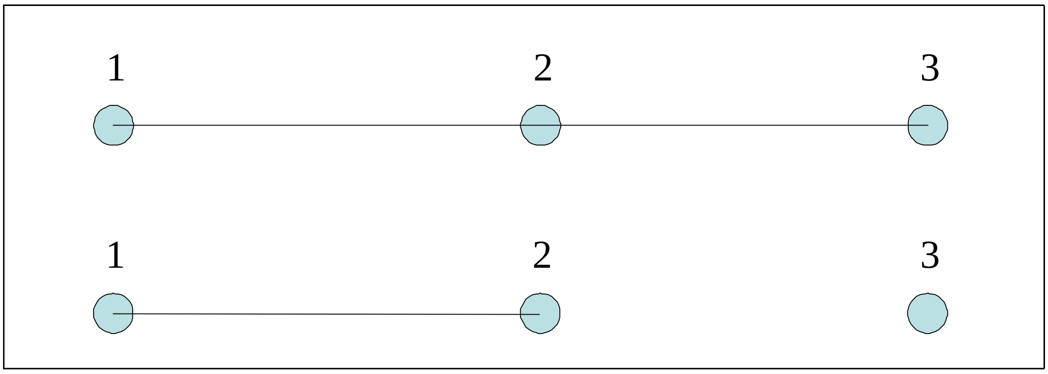

FIGURE 1. A simple network

It is plausible that the payoffs are zero for all players in case of the network linking only players 1 and 2 (lower part of the figure). After all, the two productive players cannot cooperate.

Starting with the upper network and assuming that the link between players 2 and 3 can be formed (or dissolved) by mutual consent only, the removal of the link should harm both players equally:

$$\underbrace{x_2 - 0}_{\substack{\text{loss to player 2}\\ \text{if link is removed}}} = \underbrace{x_3 - 0}_{\substack{\text{loss to player 3}\\ \text{if link is removed}}} . \quad \text{(I.6)}$$

Recognizing the symmetry between players 1 and 3 (both are productive and both need player 2 to realize their productive potential), we obain

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 \\ &= x_3 + x_3 + x_3 \ (x_1 = x_3 \text{ and eq. I.6}) \end{aligned}$$

and hence

$$(x_1, x_2, x_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

5. Cooperative and noncooperative game theories

It is sometimes suggested that non-cooperative game theory is more fundamental than cooperative game theory. Indeed, from an economic or sociological point of view, cooperative game theory seems odd in that it does not model people who "act", "know about things", or "have preferences". In cooperative game theory, people just get payoffs. Cooperative game theory is payoff-centered game theory. Noncooperative game theory (which turns around strategies and equilibria) could be termed action-centered or strategy-centered. Of course, non-cooperative game theory's strength does not come without a cost. The modeller is forced to specify in detail (sequences of) actions, knowledge and preferences. More often than not, these details cannot be obtained by the modeller. Cooperative game theory is better at providing a bird's eye view.

On the other hand, cooperative game theory is more demanding in terms of interpretation. Normally, a coalition function comes with a story describing the relationship between people. Therefore, while cooperative game theory yields payoffs, these payoffs often suggest actions.

While the two theories rely on very different ideas, they get close for two different reasons. Imagine a cooperative solution concept that produces certain payoffs for the players. One can ask the question whether a noncooperative model exists that leads to the same payoffs. This is the so-called Nash program. Of course, the inverse is also possible. Take a noncooperative model that leads to certain payoffs in equilibrium. Is there a cooperative model that also produces these payoffs?

Second, for some applications, mixtures of noncooperative and cooperative models prove quite useful. The first part of the model is noncooperative and the last cooperative. In this book, we will employ mixed models several times.

6. This book

6.1. Overview. In this book, we deal with most topics alluded to in the previous section. I finally decided on the following order:

- Part B is a careful and slow introduction into cooperative game theory. It focuses on the Shapley value and the core – the two main concepts. the Banzhaf solution is also examined. We also present a wide range of micro models through the lens of the Pareto principle which is one the most welknown cooperative solution concept.
- Parts **??** and **??** introduce additional structure on the player set. Part **??** deals with partitional values based on the Shapley value such as the AD value, the union value and the outside-option values.
  - Chapter **??** deals with partitions where the players within a component share the component's worth while outside options are taken into account. The glove game has been considered above. Also, we consider the power of parties within government coalitions. Here some political parties work together to create power. The outside options concern other parties with which alternative government coalitions could have been formed.
  - Working togehter to create worth is the reason for forming components in chapter **??**. In contrast, forming bargaining groups is the topic treated in chapter **??**. Unions are a prime example.

  - In chapter **??**, we present an application that rests on dealing with worth-creating components (firms) and bargaining components (unions) at the same time. We consider the question of how unions and unemployment benefits influence emoployment.
- Part **??** concentrates on networks and the Myerson value. Applications concern the Granovetter thesis (that weak links are more important than strong links) and hierarchies within firms.
- While the first three parts of the book deal with the payoffs obtainable by players who produce or bargain, the fourth part shifts attention to payoffs that players obtain for reasons of solidarity or by force.
- Players in parts B to **??** are atomic (indivisible). Part **??** is concerned with quite diverse models where players work part-time (chapter **??**) or "grow" in the sense of growth theory (chapter **??**) or in the sense of evolutionary theory (chapter **??**). We work with non-atomic agents who form a continuum.
- Finally, part **??** turns to non-transferable utility. We examine the allocation of goods within the Edgeworth box and also present the Nash bargaining solution.

6.2. Alternative paths through the book.

- The classical path: parts B and **??**

  Arguably, every economist worth his salt should know the Shapley value, the core, the Banzhaf solution, the core for an exchange economy and the Nash bargaining solution. If that is all you want, stick to the classical path.
- The structured path: parts B, **??** and **??**

  If you are interested in applications involving partitions and networks, you may choose to restrict attention to chapters III and VI within part B before turning to Shapley values where players are structured in some way or other.
- ...

CHAPTER II

Pareto optimality in microeconomics

Although the Pareto principle belongs to cooperative game theory, it sheds an interesting light on many different models in microeconomics. We consider bargaining between consumers, producers, countries in international trade, and bargaining in the context of public goods and externalities. We can also subsume the profit maximization and household theory under this heading. It turns out that it suffices to consider three different cases with many subcases:

- equality of marginal rates of substitution
- equality of marginal rates of transformation and
- equality of marginal rate of substitution and marginal rate of transformation

Thus, we consider a wide range of microeconomic topics through the lense of Pareto optimality.

1. Introduction: Pareto improvements

Economists are somewhat restricted when it comes to judgements on the relative advantages of economic situations. The reason is that ordinal utility does not allow for comparison of the utilities of different people.

However, situations can be ranked according to their Pareto efficiency (Vilfredo Pareto, Italian sociologue, 1848-1923). Situation 1 is called a Pareto superior to situation 2 if no individual is worse off in the first than in the second while at least one individual is strictly better off. Then, the move from 1 to 2 is called a Pareto improvement. Situations are called Pareto efficient, Pareto optimal or just efficient if Pareto improvements are not possible.

Exercise II.1. *Define Pareto optimality by way of Pareto improvements.*

Exercise II.2. *a) Is the redistribution of wealth a Pareto improvement if it reduces social inequality?*
b) Can a situation be efficient if one individual possesses everything?

This chapter rests on the premise that bargaining leads to an efficient outcome under ideal conditions. As long as Pareto improvements are available, there is no reason (so one could argue) not to "cash in" on them.

2. Identical marginal rates of substitution

2.1. Exchange Edgeworth box.

2.1.1. *Introducing the Edgeworth box for two consumers.* We consider agents or households that consume bundles of goods. A distribution of such bundles among all households is called an allocation. In a two-agent two-good environment, allocations can be visualized via the Edgeworth box. Exchange Edgeworth boxes allow to depict preferences by the use of indifference curves.

The analysis of bargaining between consumers in an exchange Edgeworth box is due to Francis Ysidro Edgeworth (1881) (1845-1926). Edgeworth is the author of a book with the beautiful title "Mathematical Psychics". Fig. 1 represents the exchange Edgeworth box for goods 1 and 2 and individuals A and B. The exchange Edgeworth box exhibits two points of origin, one for individual A (bottom left corner) and another one for individual B (top right).

Every point in the box denotes an allocation: how much of each good belongs to which individual. One possible allocation is the (initial) endowment. For all allocations $\left(x^A, x^B\right)$ with $x^A = \left(x_1^A, x_2^A\right)$ for individual A and $x^B = \left(x_1^B, x_2^B\right)$ for individual B we have

$$\begin{aligned} x_1^A + x_1^B &= \omega_1^A + \omega_1^B \text{ and} \\ x_2^A + x_2^B &= \omega_2^A + \omega_2^B. \end{aligned}$$

Individual A possesses an endowment $\omega^A = \left(\omega_1^A, \omega_2^A\right)$, i.e., ω_1^A units of good 1 and ω_2^A units of good 2. Similarly, individual B has an endowment $\omega^B = \left(\omega_1^B, \omega_2^B\right)$.

EXERCISE II.3. *Do the two individuals in fig. 1 possess the same quantities of good* 1*, i.e., do we have* $\omega_1^A = \omega_1^B$*?*

EXERCISE II.4. *Interpret the length and the breadth of the Edgeworth box!*

2.1.2. *Equality of the marginal rates of substitution.* Seen from the respective points of origin, the Edgeworth box depicts the two individuals' preferences via indifference curves. Refer to fig. 1 when you work on the following exercise.

EXERCISE II.5. *Which bundles of goods does individual A prefer to his endowment? Which allocations do both individuals prefer to their endowments?*

The two indifference curves in fig. 1, crossing at the endowment point, form the so-called exchange lens which represents those allocations that are Pareto improvements to the endowment point. A Pareto efficient allocation

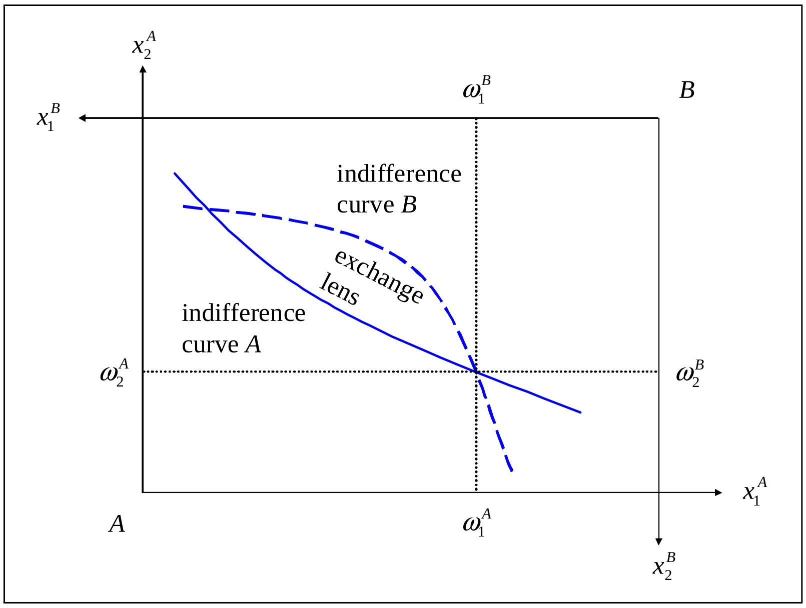

FIGURE 1. The exchange Edgeworth box

is achieved if no further improvement is possible. Then, no individual can be made better off without making the other worse off. Oftentimes, we imagine that individuals achieve a Pareto efficient point by a series of exchanges. As long as a Pareto optimum has not been reached, they will try to improve their lots.

EXERCISE II.6. *Sketch an inequitable Pareto optimum in an exchange Edgeworth box. Is the relation "allocation x is a Pareto improvement over allocation y" complete (see definition* **??**, *p.* **??***)?*

Finally, we turn to the equality of the marginal rates of substitution. Consider an exchange economy with two individuals A and B where the marginal rate of substitution of individual A is smaller than that of individual B:

$$(3 =) \left|\frac{dx_2^A}{dx_1^A}\right| = MRS^A < MRS^B = \left|\frac{dx_2^B}{dx_1^B}\right| (= 5)$$

We can show that this situation allows Pareto improvements. Individual A is prepared to give up a small amount of good 1 in exchange for at least MRS^A units (3, for example) of good 2. If individual B obtains a small amount of good 1, he is prepared to give up MRS^B (5, for example) or less units of good 2. Thus, if A gives one unit of good 1 to B, by $MRS^A < MRS^B$ individual B can offer more of good 2 in exchange than individual A would require for compensation. The two agents might agree on 4 units so that both of them would be better off. Thus, the above inequality signals the possibility of mutually beneficial trade.

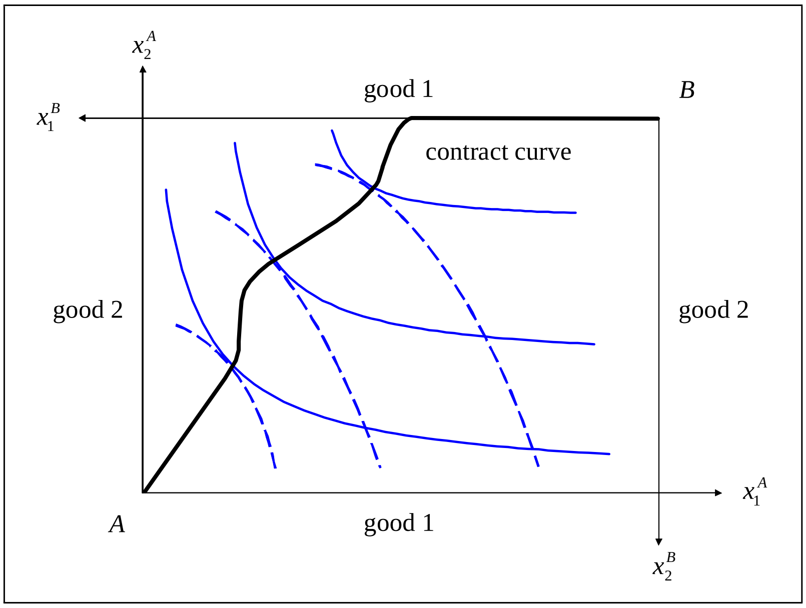

FIGURE 2. The contract curve

Differently put, Pareto optimality requires the equality of the marginal rates of substitution for any two agents A and B and any pair of goods 1 and 2. The locus of all Pareto optima in the Edgeworth box is called the contract curve or exchange curve (see fig. 2).

2.1.3. *Deriving the utility frontier.* The contract curve can be transformed into the so-called utility frontier which is nothing but the exchange Edgeworth box' equivalent of the transformation curve (production-possibility frontier) known from the production Edgeworth box. Fig. 3 shows how to construct this curve. Take point R in the upper part of figure 3. Here, individual A achieves his utility level U_R^A. Since R is a Pareto efficient point on the contract curve, it is not possible for individual B to achieve a higher level of utility than U_R^B. The pair of utility levels $\left(U_R^A, U_R^B\right)$ is depicted in the lower part. In a similar fashion, point T (upper part) is transformed into point T (lower part). The resulting curve is called utility frontier. Given some utility level of individual A, this curve represents the maximal utility level possible for individual B.

EXERCISE II.7. *Are points S and T in fig. 4 Pareto efficient?*

EXERCISE II.8. *Two consumers meet on an exchange market with two goods. Both have the utility function $U(x_1, x_2) = x_1 x_2$. Consumer A's endowment is $(10, 90)$, consumer B's is $(90, 10)$.*
a) Depict the endowments in the Edgeworth box!
b) Find the contract curve and draw it!
c) Find the best bundle that consumer B can achieve through exchange!

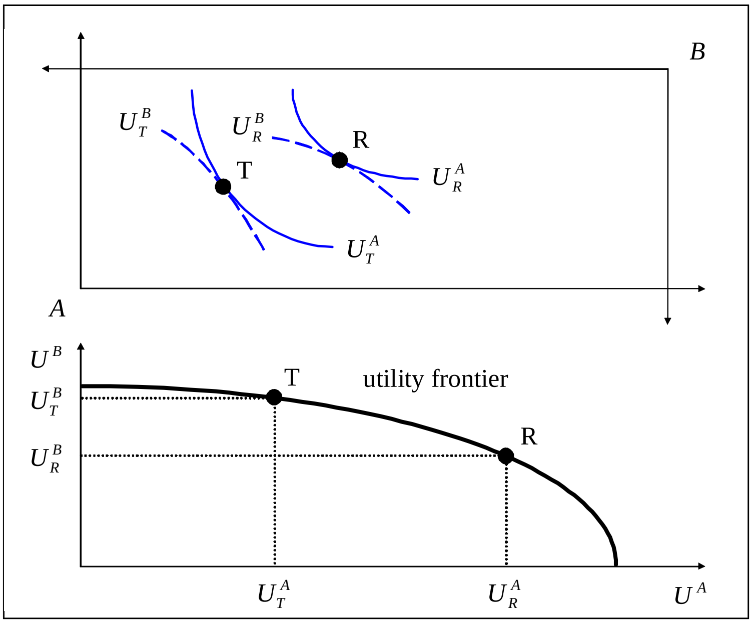

FIGURE 3. Construction of the utility frontier

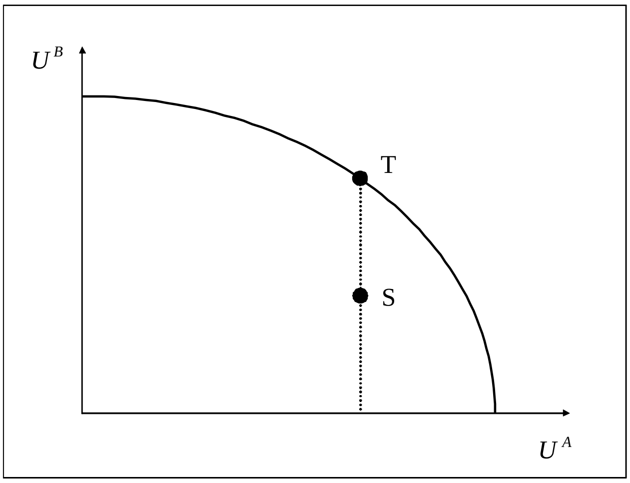

FIGURE 4. Utility-possibility curve

d) Draw the Pareto improvement (exchange lens) and the Pareto-efficient Pareto improvements!
e) Sketch the utility frontier!

2.2. Production Edgeworth box. With respect to the production Edgeworth box, we can argue in a similar fashion. We have denoted the marginal willingness to pay for input factors used by a producer 1 for an additional unit of labor L in terms of capital C as the marginal rate of technical substitution, $MRTS_1 = \left|\frac{dC_1}{dL_1}\right|$. If two producers 1 and 2 produce goods 1 and 2, respectively, with inputs labor and capital, both can increase their production as long as the marginal rates of technical substitution differ. Thus, Pareto efficiency means

$$\left|\frac{dC_1}{dL_1}\right| = MRTS_1 \stackrel{!}{=} MRTS_2 = \left|\frac{dC_2}{dL_2}\right|$$

so that the marginal willingness to pay for input factors are the same.

2.3. Two markets – one factory. The third subcase under the heading "equality of the marginal willingness to pay" concerns a firm that produces in one factory but supplies two markets 1 and 2. The idea is to consider the marginal revenue $MR = \frac{dR}{dx_i}$ as the monetary marginal willingness to pay for selling one extra unit of good i. How much can a firm pay for the sale of one additional unit?

Thus, the marginal revenue is a marginal rate of substitution $\left|\frac{dR}{dx_i}\right|$. The role of the denominator good is taken over by good 1 or 2, respectively, while the nominator good is "money" (revenue). Now, profit maximization by a firm selling on two markets 1 and 2 implies

$$\left|\frac{dR}{dx_1}\right| = MR_1 \stackrel{!}{=} MR_2 = \left|\frac{dR}{dx_2}\right|$$

as we have seen in chapter **??**, p. **??**.

2.4. Two firms (cartel). The monetary marginal willingness to pay for producing *and* selling one extra unit of good y is a marginal rate of substitution where the denominator good is good 1 or 2 while the nominator good represents "money" (profit). We know from chapter **??** (p. **??**) that two cartelists 1 and 2 producing the quantities x_1 and x_2, respectively, maximize their joint profit

$$\Pi_{1,2}(x_1, x_2) = \Pi_1(x_1, x_2) + \Pi_2(x_1, x_2)$$

by obeying the first-order conditions

$$\frac{\partial \Pi_{1,2}}{\partial x_1} \stackrel{!}{=} 0 \stackrel{!}{=} \frac{\partial \Pi_{1,2}}{\partial x_2}$$

so that their marginal rates of substitution are the same when profit is understood as joint profit. If $\frac{\partial \Pi_{1,2}}{\partial x_2}$ were higher than $\frac{\partial \Pi_{1,2}}{\partial x_1}$ the cartel could increase profits by shifting the production of one unit from firm 1 to firm 2.

Note that a similar condition holds for the Cournot equilibrium (see p. **??**),

$$\frac{\partial \Pi_1}{\partial x_1} \stackrel{!}{=} 0 \stackrel{!}{=} \frac{\partial \Pi_2}{\partial x_2}.$$

However, this is definitely *not* an example for Pareto optimality (although two marginal rates of substitution coincide). Rather, for each individual firm, it is an example of Pareto optimality where a marginal rate of substitution equals a marginal rate of transformation (see below subsection 4.4, p. 22).

3. Identical marginal rates of transformation

3.1. Two factories – one market. While the marginal revenue can be understood as the monetary marginal willingness to pay for selling, the marginal cost $MC = \frac{dC}{dy}$ can be seen as the monetary marginal opportunity cost of production. How much money (the second good) must the producer forgo in order to produce an extra unit of y (the first good)? Thus, the margincal cost can be seen as a special case of the marginal rate of transformation, $MRT = \left|\frac{dx_2}{dx_1}\right|^{\text{transformation curve}}$.

According to chapter **??**, p. **??**, a firm supplying a market from two factories (or a cartel in case of homogeneous goods), obeys the equality

$$MC_1 \overset{!}{=} MC_2.$$

The cartel also makes clear that Pareto improvements and Pareto optimality have to be defined relative to a specific group of agents. While the cartel solution (maximizing the sum of profits) can be optimal for the producers, it is not for the economy as a whole because the sum of producers' and consumers' (!) rent may well be below the welfare optimum.

3.2. Bargaining between countries (international trade). David Ricardo (1772–1823) has shown that international trade is profitable as long as the rates of transformation between any two countries are different. Let us consider the classic example of England and Portugal producing wine (W) and cloth (Cl). Suppose that the marginal rates of transformation differ:

$$4 = MRT^P = \left|\frac{dW}{dCl}\right|^P > \left|\frac{dW}{dCl}\right|^E = MRT^E = 2.$$

In that case, international trade is Pareto-improving. Indeed, let England produce another unit of cloth Cl that it exports to Portugal. England's production of wine reduces by $MRT^E = 2$ gallons. Portugal, that imports one unit of cloth, reduces the cloth production and can produce additional $MRT^P = 4$ units of wine. Therefore, if England obtains 3 gallons of wine in exchange for the one unit of cloth it gives to Portugal, both countries are better off.

Ricardo's theorem is known under the heading of "comparative cost advantage". So far, it is unclear why this is a good name for his theorem. The answer is provided by the following

LEMMA II.1. *Assume that f is a differentiable transformation function $x_1 \mapsto x_2$. Assume also that the cost function $C(x_1, x_2)$ is differentiable. Then, the marginal rate of transformation between good 1 and good 2 can be obtained by*

$$MRT(x_1) = \left|\frac{df(x_1)}{dx_1}\right| = \frac{MC_1}{MC_2}.$$

PROOF. Reconsider the production Edgeworth box encountered in chapter ??. We assume a given volume of factor endowments and (perfect competition!) given factor prices. Then, the overall cost for the production of goods 1 and 2 is constant and does not change along the transformation curve. Therefore, we can write

$$C(x_1, x_2) = C(x_1, f(x_1)) = \text{constant}.$$

If, now, we produce more of good 1 and less of good 2, the costs do not change:

$$\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}\frac{df(x_1)}{dx_1} = 0.$$

Solving for the marginal rate of transformation yields

$$MRT = -\frac{df(x_1)}{dx_1} = \frac{MC_1}{MC_2}.$$

□

Now we have Ricardo's result in the form it is usually presented: As long as the comparative costs (more precise: the ratio of marginal costs) between two goods differ, international trade is worthwhile for both countries.

Thus, Pareto optimality requires the equality of the marginal opportunity costs between any two goods produced in any two countries. The economists before Ricardo clearly saw that absolute cost advantages make international trade profitable. If England can produce cloth cheaper than Portugal while Portugal can produce wine cheaper than England, we have

$$\begin{aligned} MC_{Cl}^E &< MC_{Cl}^P \text{ and} \\ MC_W^E &> MC_W^P \end{aligned}$$

so that England should produce more cloth and Portugal should produce more wine. Ricardo observed that for the implied division of labor to be profitable, it is sufficient that the ratio of the marginal costs differ:

$$\frac{MC_{Cl}^E}{MC_W^E} < \frac{MC_{Cl}^P}{MC_W^P}.$$

Do you see that this inequality follows from the two inequalities above, but not vice versa?

4. Equality between marginal rate of substitution and marginal rate of transformation

4.1. Base case. Imagine two goods consumed at a marginal rate of substitution MRS and produced at a marginal rate of transformation MRT. We now show that optimality also implies $MRS = MRT$. Assume, to the contrary, that the marginal rate of substitution (for a consumer) is lower than the marginal rate of transformation (for a producer):

$$MRS = \left|\frac{dx_2}{dx_1}\right|^{\text{indifference curve}} < \left|\frac{dx_2}{dx_1}\right|^{\text{transformation curve}} = MRT.$$

If the producer reduces the production of good 1 by one unit, he can increase the production of good 2 by MRT units. The consumer has to renounce the one unit of good 1, and he needs at least MRS units of good 2 to make up for this. By $MRT > MRS$ the additional production of good 2 (come about by producing one unit less of good 1) more than suffices to compensate the consumer. Thus, the inequality of marginal rate of substitution and marginal rate of transformation points to a Pareto-inefficient situation.

4.2. Perfect competition. We want to apply the formula

$$MRS \overset{!}{=} MRT$$

to the case of perfect competition. For the output space, we have the condition

$$p \overset{!}{=} MC.$$

We have derived "price equals marginal cost" as the profit-maximizing condition on p. **??** and have discussed the welfare-theoretic implications on p. **??**.

We can link the two formulae by letting good 2 be money with price 1.

- Then, the marginal rate of substitution tells us the consumer's monetary marginal willingness to pay for one additional unit of good 1 (see pp. **??**). Cum grano salis, the price can be taken to measure this willingness to pay for the marginal consumer (the last consumer prepared to buy the good).
- The marginal rate of transformation is the amount of money one has to forgo for producing one additional unit of good 1, i.e., the marginal cost.

Therefore, we obtain

$$\text{price} = \text{marginal willingness to pay} \overset{!}{=} \text{marginal cost}.$$

In a similar fashion, we can argue for inputs. The marginal value product $MVP = p\frac{dy}{dx}$ is the monetary marginal willingness to pay for the factor use while the factor price w can be understood as the monetary marginal opportunity cost of employing the factor. Thus, we reobtain the optimization condition for a price taker on both the input and the output market introduced in p. **??**:

$$\text{marginal value product} \overset{!}{=} \text{factor price}.$$

4.3. First-degree price discrimination. The Cournot monopoly clearly violates the "price equals marginal cost" rule. However, first-degree price discrimination fulfills this rule as shown on pp. **??**.

4.4. Cournot monopoly. A trivial violation of Pareto optimality ensues if a single agent acts in a non-optimal fashion. Just consider consumer and producer as a single person. For the Cournot monopolist, the $MRS \overset{!}{=} MRT$ formula can be rephrased as the equality between

- the monetary marginal willingness to pay for selling – this is the marginal revenue $MR = \frac{dR}{dy}$ (see obve p. 18) – and
- the monetary marginal opportunity cost of production, the marginal cost $MC = \frac{dC}{dy}$ (p. 19).

4.5. Household optimum. A second violation of efficiency concerns the consuming household. It "produces" goods by using his income to buy them, $m = p_1 x_1 + p_2 x_2$ in case of two goods.

EXERCISE II.9. *The prices of two goods* 1 *and* 2 *are* $p_1 = 6$ *and* $p_2 = 2$*, respectively. If the household consumes one additional unit of good* 1*, how many units of good* 2 *does he have to renounce?*

The exercise helps us understand that the marginal rate of transformation is the price ratio,

$$MRT = \frac{p_1}{p_2},$$

that we also know under the heading of "marginal opportunity cost". (Alternatively, consider the transformation function $x_2 = f(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$.). Seen this way, $MRS \overset{!}{=} MRT$ is nothing but the famous condition for household optimality dervied on pp. **??**.

4.6. External effects and the Coase theorem.

4.6.1. *External effects and bargaining.* The famous Coase theorem can also be interpreted as an instance of $MRS \overset{!}{=} MRT$. We present this example in some detail.

External effects are said to be present if consumption or production activities are influenced positively or negatively while no compensation is paid for this influence. Environmental issues are often discussed in terms of negative externalities. Also, the increase of production exerts a negative influence on other firms that try to sell subsitutes. Reciprocal effects exist between beekeepers and apple planters.

Consider a situation where A pollutes the environment doing harm to B. In a very famous and influential paper, Coase (1960) argues that economists have seen environmental and similar problems in a misguided way.

First of all, externalities are a “reciprocal problem”. By this Coase means that restraining A from polluting harms A (and benefits B). According to Coase, the question to be decided is whether the harm done to B (suffering the polluting) is greater or smaller than the harm done to A (by stopping A's polluting activities).

Second, many problems resulting from externalities stem from missing property rights. Agent A may not be in a position to sell or buy the right to pollute from B simply because property exists for cars and real estate but not for air, water or quietness. Coase suggests that the agents A and B bargaing about the externality. If, for example, A has the right to pollute (i.e., is not liable for the damage cause by him), B can give him some money so that A reduce his harmful (to B) activity. If B has the right not to suffer any pollution (i.e., A is liable), A could approach B and offer some money in order to pursue some of the activity benefitting him. Coase assumes (as we have done in this chapter) that the two parties bargain about the externality so as to obtain a Pareto-efficient outcome.

The Nobel prize winner (of 1991) presents a startling thesis: the externality (the pollution etc.) is independent on the initial distribution of property rights. This thesis is also known as the invariance hypothesis.

4.6.2. *Straying cattle.* Coase (1960) discusses the example of a cattle raiser and a crop farmer who possess adjoining land. The cattle regularly destroys part of the farmer's crop. In particular, consider the following table:

number of steers	marginal profit	marginal crop loss
1	4	1
2	3	2
3	2	3
4	1	4

The cattle raiser's marginal profit from steers is a decreasing function of the number of steers while the marginal crop loss increases. Let us begin with the case where the cattle raiser is liable. He can pay the farmer up to 4 (thousand Euros) for allowing him to have one cattle destroy crop. Since the farmer's compensating variation is 1, the two can easily agree on a price of 2 or 3.

The farmer and cattle raiser will also agree to have a second steer roam the fields, for a price of $2\frac{1}{2}$. However, there are no gains from trade to be had for the third steer. The willingness to pay of 2 is below the compensation money of 3.

If the cattle raiser is not liable, the farmer has to pay for reducing the number of steers from 4 to 3. A Pareto improvement can be have for any price between 1 and 4. Also, the farmer will convince the cattle raiser to take the third steer, but not the second one, off the field.

Thus, Coase seems to have a good point – irrespective of the property rights (the liability question), the number of steers and the amount of crop damaged is the same.

The reason for the validity (so far) of the Coase theorem is the fact that forgone profits are losses and forgone losses are profits. Therefore, the numbers used in the comparisons are the same.

It is about time to tell the reader why we talk about the Coase theorem in the $MRS \overset{!}{=} MRT$ section. From the cartel example, we are familiar with the idea of finding a Pareto optimum by looking at joint profits. We interpret the cattle raiser's marginal profit as the (hypothetical) joint firm's willingness to pay for another steer and the marginal crop loss incurred by the farmer as the joint firm's marginal opportunity cost for that extra steer.

We close this section by throwing in two caveats:

- If consumers are involved, the distribution of property rights has income effects. Then, Coase's theorem does not hold any more (see Varian 2010, chapter 31).
- More important is the objection raised by Wegehenkel (1980). The distribution of property rights determine who pays whom. Thus, if the property rights were to change from non-liability to liability, cattle raising becomes a less profitable business while growing crops is more worthwhile as before. In the medium run, agents will move to the profitable occupations with effects on the crop losses (the sign is not clear a priori).

4.7. Public goods. Public goods are defined by non-rivalry in consumption. While an apple can be eaten only once, the consumption of a public good by one individual does not reduce the consumption possibilities by others. Often-cited examples include street lamps or national defence.

Consider two individuals A and B who consume a private good x (quantities x^A and x^B, respectively) and a public good G. The optimality condition is

$$\begin{aligned} & MRS^A + MRS^B \\ = \; & \left| \frac{dx^A}{dG} \right|^{\text{indifference curve}} + \left| \frac{dx^B}{dG} \right|^{\text{indifference curve}} \\ \overset{!}{=} \; & \left| \frac{d\left(x^A + x^B\right)}{dG} \right|^{\text{transformation curve}} = MRT. \end{aligned}$$

Assume that this condition is not fulfilled. For example, let the marginal rate of transformation be smaller than the sum of the marginal rates of substitution. Then, it is a good idea to produce one additional unit of the public good. The two consumers need to forgo MRT units of the private good. However, they are prepared to give up $MRS^A + MRS^B$ units of the private good in exchange for one additional unit of the public good. Thus, they can give up more than they need to. Assuming monotonicity, the two consumers are better off than before and the starting point (inequality) does not characterize a Pareto optimum.

Once more, we can assume that good x is the numéraire good (money with price 1). Then, the optimality condition simplifies and Pareto efficiency requires that the sum of the marginal willingness' to pay equals the marginal cost of the public good.

EXERCISE II.10. *In a small town, there live 200 people* $i = 1, ..., 200$ *with identical preferences. Person i's utility function is* $U_i(x_i, G) = x_i + \sqrt{G}$*, where* x_i *is the quantity of the private good and* G *the quantity of the public good. The prices are* $p_x = 1$ *and* $p_G = 10$*, respectively. Find the Pareto-optimal quantity of the public good.*

Thus, by the non-rivalry inconsumption, we do not quite get a subrule of $MRS \stackrel{!}{=} MRT$ but something similar.

5. Topics and literature

The main topics in this chapter are

- Pareto efficiency
- Pareto improvement
- exchange Edgeworth box
- contract curve
- exchange lense
- core
- international trade
- external effects
- quantity cartel
- public goods
- first-degree price discrimination

We recommend the textbook by

6. Solutions

Exercise II.1

A situation is Pareto optimal if no Pareto improvement is possible.

Exercise II.2

a) A redistribution that reduces inequality will harm the rich. Therefore, such a redistribution is not a Pareto improvement.

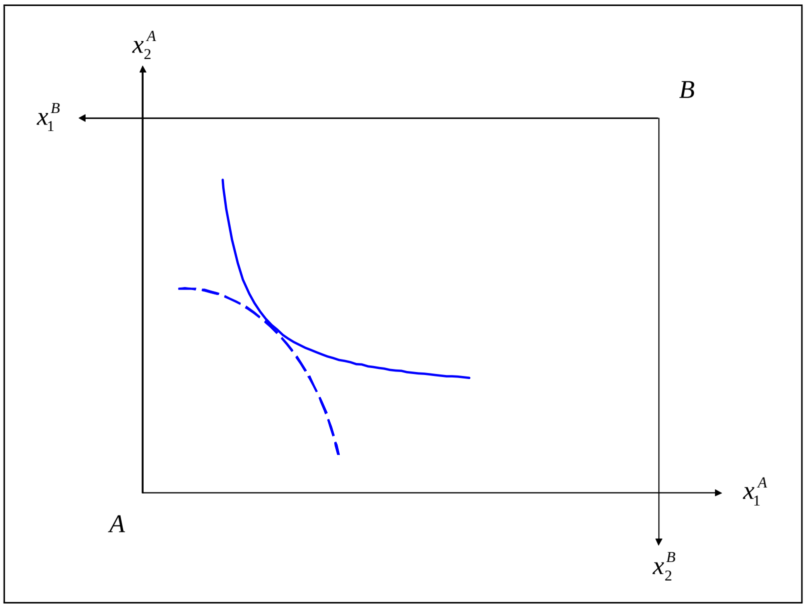

FIGURE 5. Pareto optimality and equality

b) Yes. It is not possible to improve the lot of the have-nots without harming the individual who possesses everything.

Exercise II.3

No, obviously ω_1^A is much larger than ω_1^B.

Exercise II.4

The length of the exchange Edgeworth box represents the units of good 1 to be divided between the two individuals, i.e., the sum of their endowment of good 1. Similarly, the breadth of the Edgeworth box is $\omega_2^A + \omega_2^B$.

Exercise II.5

Individual A prefers all those bundels x_A that lie to the right and above the indifference curve that crosses his endowment point. The allocations preferred by both individuals are those in the hatched part of fig. 1.

Exercise II.6

For the first question, you should have drawn something like fig. 5. Fig. 6 makes clear that we can have bundles A and B where A is no Pareto improvement over B and B is no improvement over A. Thus, the relation is not complete.

Exercise II.7

Point S is not Pareto optimal. At T, individual B is better off while A's utility level is the same as at S. From T no Pareto improvement is possible. Therefore, T is Pareto optimal.

Exercise II.8

a) See fig. 7,

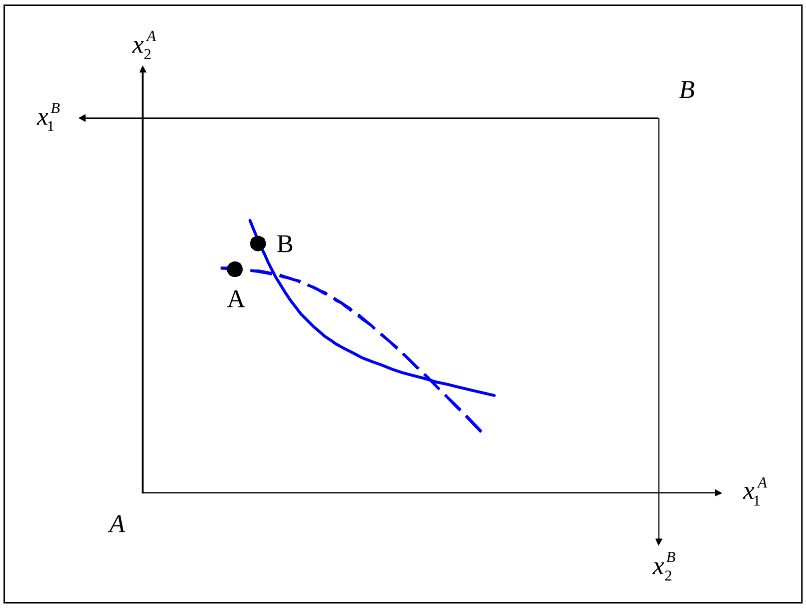

FIGURE 6. Incompleteness

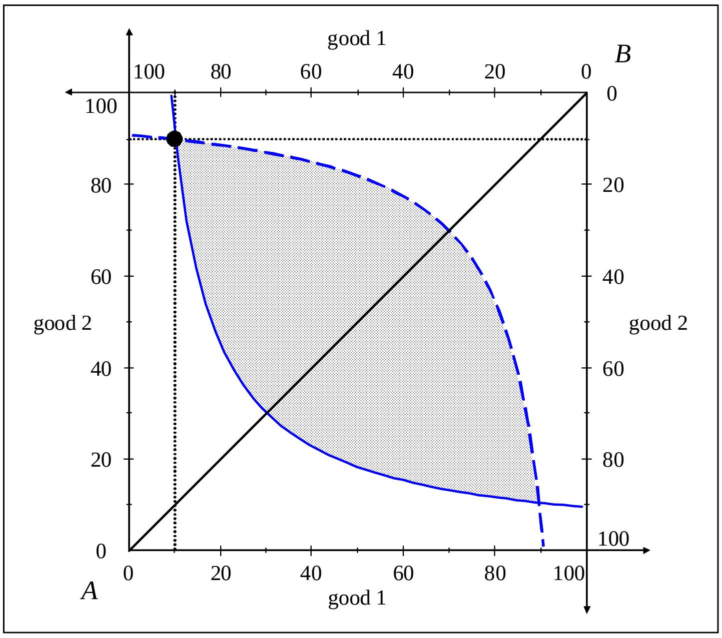

FIGURE 7. The answer to parts a) and d)

b) $x_1^A = x_2^A$,
c) $(70, 70)$.
d) The exchange lens is dotted in fig. 7. The Pareto efficient Pareto improvements are represented by the contract curve within this lens.
e) The utility frontier is downward sloping and given by $U_B(U_A) = \left(100 - \sqrt{U_A}\right)^2$.

Exercise II.9

If the household consumers one additional unit of good 1, he has to pay Euro 6. Therefore, he has to renounce 3 units of good 2 that also cost Euro 6 = Euro 2 times 3.

Exercise II.10

The marginal rate of transformation $\left|\frac{d(\sum_{i=1}^{200} x_i)}{dG}\right|$ equals $\frac{p_G}{p_x} = \frac{10}{1} = 10$. The marginal rate of substitution for inhabitant i is

$$\left|\frac{dx^i}{dG}\right|^{\text{indifference curve}} = \frac{MU_G}{MU_{x^i}} = \frac{\frac{1}{2\sqrt{G}}}{1} = \frac{1}{2\sqrt{G}}.$$

Applying the optimality condition yields

$$200 \cdot \frac{1}{2\sqrt{G}} \overset{!}{=} 10$$

and hence $G = 100$.

7. Further exercises without solutions

Agent A has preferences on (x_1, x_2), that can be represented by $u^A(x_1^A, x_2^A) = x_1^A$. Agent B has preferences, which are represented by the utility function $u^B(x_1^B, x_2^B) = x_2^B$. Agent A starts with $\omega_1^A = \omega_2^A = 5$, and B has the initial endowment $\omega_1^B = 4, \omega_2^B = 6$.

(a) Draw the Edgeworth box, including
  - ω,
  - an indifference curve for each agent through ω!

(b) Is $(x_1^A, x_2^A, x_1^B, x_2^B) = (6, 0, 3, 11)$ a Pareto-improvement compared to the initial allocation?

(c) Find the contract curve!

The Shapley value and the core

Part B

The Shapley value and the core

The first part of our course explains some important basic concepts for transferable utility without additional structure. Chapter III introduces Pareto efficiency, the Shapley value and the core for a simple game, the gloves game. We present many examples of cooperative games in chapter IV. Games can be understood as vectors – this is the point of view we mention in the following chapter and discuss in detail in chapter V. We then deal with the axiomatization of the Shapley value in chapter VI. Finally, the Banzhaf valus is treated in chapter 7. Partitions and graphs have no role to play in this part of the book.

CHAPTER III

The gloves game

1. The coalition function

In this chapter, we concentrate on a particular game, the gloves game. Some players have a left glove and others a right glove. Single gloves have a worth of zero while pairs have a worth of 1 (Euro). The coalition function for the gloves game is given by

$$\begin{aligned} v_{L,R} &: 2^N \to \mathbb{R} \\ & K \mapsto v_{L,R}(K) = \min(|K \cap L|, |K \cap R|), \end{aligned}$$

where

- N is the set of players (also called the grand coalition),
- L the set of players holding a left glove and R the set of right-glove owners together with $L \cap R = \emptyset$ and $L \cup R = N$,
- $v_{L,R}$ denotes the coalition function for the gloves game,
- 2^N stands for N's power set, i.e., the set of all subsets of N (the domain of $v_{L,R}$),
- $\mathbb{R}$ is the set of real numbers (the range of $v_{L,R}$),
- K is a coalition, i.e., d.h. $K \subseteq N$ or $K \in 2^N$,
- $|K|$ means the number of elements (players) in K and
- $\min(x, y)$ is the smallest of the two numbers x and y.

Thus, the coalition function $v_{L,R}$ attributes the number of pairs in possession of some coalition K to that coalition. Reminding the reader of the symbol $\emptyset$ for the empty set (with obeys $|\emptyset| = 0$, of course), we present

Definition III.1 (player sets and coalition functions). *Player sets and coalition functions are specified by the following definitions:*

- *Finite and nonempty player sets are denoted by N. More often than not, we have $N = \{1, ..., n\}$ with $n \in \mathbb{N}$.*
- *$v : 2^N \to \mathbb{R}$ is called a coalition function if v fulfills $v(\emptyset) = 0$. $v(K)$ is called coalition K's worth.*
- *For any given coalition function v, its player set can be addressed by $N(v)$ or, more simply, N.*
- *We denote the set of all games on N by $V(N)$ and the set of all games (for any player set N) by V.*

Exercise III.1. *Assume* $N = \{1,2,3,4,5\}$, $L = \{1,2\}$ *and* $R = \{3,4,5\}$. *Find the worths of the coalitions* $K = \{1\}$, $K = \emptyset$, $K = N$ *and* $K = \{2,3,4\}$. *Hint:* $\emptyset$ *is the empty set with obeys* $|\emptyset| = 0$.

The above exercise makes clear that $v_{L,R}$ is, indeed, a coalition function. The requirement of $v(\emptyset) = 0$ makes perfect sense: a group of zero agents cannot achieve anything.

Exercise III.2. *Which of the following propositions make sense? Any coalition* K *and any grand coalition* N *fulfill*

- $K \in N$ *and* $K \in 2^N$,
- $K \subseteq N$ *and* $K \subseteq 2^N$,
- $K \in N$ *and* $K \subseteq 2^N$ *and/or*
- $K \subseteq N$ *and* $K \in 2^N$?

In this book, we focus on transferable utility where v attaches a real number to all coalitions. $v(K)$ is the worth or the utility sum created by the members from K. The basic idea is to distribute $v(K)$ or $v(N)$ among the members from K or N. Thus, the utility is "transferable".

Transferability is a serious assumption and does not work well in every model. We need non-transferable utility for the analysis of exchange within an Edgeworth box. Transferable utility is justfied if utility can be measured in terms of money and if the agents are risk neutral.

We can interpret the gloves game as a market game where the left-glove owners form one market side and the right-glove owners the other. We need to distinguish the worth (of a coalition) from the payoff acrruing to players.

2. Summing and zeros

Payoffs for players are summarized in payoff vectors:

Definition III.2. *For any finite and nonempty player set* $N = \{1,...,n\}$, *a payoff vector*

$$x = (x_1,...,x_n) \in \mathbb{R}^n$$

specifies payoffs for all players $i = 1,...,n$.

It is possible to sum coalition functions and it is possible to sum payoff vectors. Summation of vectors is easy – just sum each component individually:

Exercise III.3. *Determine the sum of the vectors*

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}!$$

If we have three players, it is obvious that the first component belongs to player 1, the second to player 2 etc. Note the difference between payoff-vector summation

$$x+y=\begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix}+\begin{pmatrix} y_1 \\ y_2 \\ \\ y_n \end{pmatrix}=\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \\ x_n+y_n \end{pmatrix}$$

and payoff summation

$$\sum_{i=1}^{n} x_i.$$

Vector summation is possible for coalition functions, too. For example, we obtain the sum $v_{\{1\},\{2,3\}}+v_{\{1,2\},\{3\}}$ by summing the worths $v_{\{1\},\{2,3\}}(K)+v_{\{1,2\},\{3\}}(K)$ for every coalition K, from the empty set $\emptyset$ down to the grand coalition $\{1,2,3\}$:

$$\begin{pmatrix} \emptyset:0 \\ \{1\}:0 \\ \{2\}:0 \\ \{3\}:0 \\ \{1,2\}:1 \\ \{1,3\}:1 \\ \{2,3\}:0 \\ \{1,2,3\}:1 \end{pmatrix}+\begin{pmatrix} \emptyset:0 \\ \{1\}:0 \\ \{2\}:0 \\ \{3\}:0 \\ \{1,2\}:0 \\ \{1,3\}:1 \\ \{2,3\}:1 \\ \{1,2,3\}:1 \end{pmatrix}=\begin{pmatrix} \emptyset:0 \\ \{1\}:0 \\ \{2\}:0 \\ \{3\}:0 \\ \{1,2\}:1 \\ \{1,3\}:2 \\ \{2,3\}:1 \\ \{1,2,3\}:2 \end{pmatrix}$$

Of course, we need to agree upon a specific order of coalitions.

Mathematically speaking, $\mathbb{R}^n$ and $V(N)$ can be considered as vector spaces. Vector spaces have a zero. The zero from $\mathbb{R}^n$ is

$$\underset{\in\mathbb{R}^n}{0}=\left(\underset{\in\mathbb{R}}{0},...,\underset{\in\mathbb{R}}{0}\right)$$

where the zero on the left-hand side is the zero vector while the zeros on the right-hand side are just the zero payoffs for all the individual players. In the vector space of coalition functions, $0\in V(N)$ is the function that attributes the worth zero to every coalition, i.e.,

$$\underset{\in V(N)}{0}(K)=\underset{\in\mathbb{R}}{0} \text{ for all } K\subseteq N$$

3. Solution concepts

For the time being, cooperative game theory consists of coalition functions and solution concepts. The task of solution concepts is to define and defend payoffs as a function of coalition functions. That is, we take a coalition function, apply a solution concept and obtain payoffs for all the players.

Solution concepts may be point-valued (solution function) or set-valued (solution correspondence). In each case, the domain is the set of all games

V for any finite player sets N. A solution function associates each game with exactly one payoff vector while a correspondence allows for several or no payoff vectors.

Definition III.3 (solution function, solution correspondence). *A function σ that attributes, for each coalition function v from V, a payoff to each of v's players,*

$$\sigma(v) \in \mathbb{R}^{|N(v)|},$$

is called a solution function (on V)[1]*. Player i's payoff is denoted by $\sigma_i(v)$. In case of $N(v) = \{1, ..., n\}$, we also write $(\sigma_1(v), ..., \sigma_n(v))$ for $\sigma(v)$ or $(\sigma_i(v))_{i \in N(v)}$.*

A correspondence that attributes a set of payoff vectors to every coalition function v,

$$\sigma(v) \subseteq \mathbb{R}^{|N(v)|}$$

is called a solution correspondence (on V).

Solution functions and solution correspondences are also called solution concepts (on V).

Ideally, solution concepts are described both algorithmically and axiomatically. An algorithm is some kind of mathematical procedure (a more less simple function) that tells how to derive payoffs from the coalition functions. Consider, for example, these four solutions concepts in algorithmic form:

- player 1 obtains $v(N)$ and the other players zero,
- every player gets 100,
- every player gets $v(N)/n$,
- every player i's payoff set is given by $[v(\{i\}), v(N)]$ (which may be the empty set).

Alternatively, solution concepts can be defined by axioms. For example, axioms might demand that

- all the players obtain the same payoff,
- no more than $v(N)$ is to be distributed among the players,
- player 1 is to get twice the payoff obtained by player 2,
- the names of players have no role to play,
- every player gets $v(N) - v(N \backslash \{i\})$.

Axioms pin down the players' payoffs, more or less. Axioms may also make contradictory demands. We present the most familiar axioms in the following sections.

[1]More formallay, a solution function on G is given by

$$\sigma : G \to \cup_{k \in \mathbb{N}} \mathbb{R}^k, \sigma(v) \in \mathbb{R}^{|N(v)|}.$$

4. Pareto efficiency

Arguably, Pareto efficiency is the single most often applied solution concept in economics – rivaled only by Nash equilibrium from noncooperative game theory. For the gloves game, Pareto efficiency is defined by

$$\sum_{i\in N} x_i = v_{L,R}(N).$$

Thus, the sum of all payoffs is equal to the number of glove pairs. It is instructive to write this equality as two inequalities:

$$\begin{aligned}\sum_{i\in N} x_i &\leq v_{L,R}(N) \text{ (feasibility) and}\\ \sum_{i\in N} x_i &\geq v_{L,R}(N) \text{ (the grand coalition cannot block } x).\end{aligned}$$

According to the first inequality, the players cannot distribute more than they (all together) can "produce". This is the requirement of feasibility.

Imagine that the second inequality were violated. Then, we have $\sum_{i=1}^n x_i < v_{L,R}(N)$ and the players would leave "money on the table". All players together could block (or contradict) the payoff vector x. This means they can propose another payoff vector that is both feasible and better for the players. Indeed, the payoff vector $y = (y_1, ..., y_n)$ defined by

$$y_i = x_i + \frac{1}{n}\left(v_{L,R}(N) - \sum_{i=1}^n x_i\right), i \in N,$$

does the trick. y improves upon x.

Exercise III.4. *Show that the payoff vector y is feasible.*

Normally, Pareto efficiency is defined by "it is impossible to improve the lot of one player without making other players worse off". If a sum of money is distributed among the player, we can also define Pareto efficiency by "it is impossible to improve the lot of all players". The additional sum of money that makes one player better off (first definition) can be spread among all the players (second definition).

Definition III.4 (feasibility and efficiency). *Let $v \in V(N)$ be a coalition function and let $x \in \mathbb{R}^n$ be a payoff vector. x is called*

- *blockable by N in case of*

$$\sum_{i=1}^n x_i < v(N),$$

- *feasible in case of*

$$\sum_{i\in N} x_i \leq v(N)$$

- *and efficient or Pareto efficient in case of*

$$\sum_{i\in N} x_i = v(N).$$

Thus, an efficient payoff vector is feasible and cannot be blocked by the grand coalition N. Obviously, Pareto efficiency is a solution correspondence, not a solution function.

EXERCISE III.5. *Find the Pareto-efficient payoff vectors for the gloves game* $v_{\{1\},\{2\}}$!

For the gloves game, the solution concept "Pareto efficiency" has two important drawbacks:

- We have very many solutions and the predictive power is weak. In particular, a left-hand glove can have any price, positive or negative.
- The payoffs for a left-glove owner does not depend on the number of left and right gloves in our simple economy. Thus, the relative scarcity of gloves is not reflected by this solution concept.

We now turn to a solution concept that generalizes the idea of blocking from the grand coalition to all coalitions.

5. The core

Pareto efficiency demands that the grand coalition should not be in a position to make all players better off. Extending this idea to all coalitions, the core consists of those feasible (!) payoff vectors that cannot be improved upon by any coalition with its own means. Formally, we have

DEFINITION III.5 (blockability and core). *Let* $v \in V(N)$ *be a coalition function. A payoff vector* $x \in \mathbb{R}^n$ *is called blockable by a coalition* $K \subseteq N$ *if*

$$\sum_{i\in K} x_i < v(K)$$

holds. The core is the set of all those payoff vectors x *fulfilling*

$$\begin{aligned} \sum_{i\in N} x_i &\leq v(N) \text{ (feasibility) and} \\ \sum_{i\in K} x_i &\geq v(K) \text{ for all } K \subseteq N \text{ (no blockade by any coalition).} \end{aligned}$$

Do you see that every payoff vector from the core is also Pareto efficient? Just take $K := N$.

The core is a stricter concept than Pareto efficiency. It demands that no coalition (not just the grand coalition) can block any of its payoff vectors. Let us consider the gloves game for $L = \{1\}$ and $R = \{2\}$. By Pareto efficiency, we can restrict attention to those payoff vectors $x = (x_1, x_2)$ that fulfill $x_1+x_2 = 1$. Furthermore, x may not be blocked by one-man coalitions:

$$\begin{aligned} x_1 &\geq v_{L,R}(\{1\}) = 0 \text{ and} \\ x_2 &\geq v_{L,R}(\{2\}) = 0. \end{aligned}$$

Hence, the core is the set of payoff vectors $x = (x_1, x_2)$ obeying

$$x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0.$$

Are we not forgetting about $K = \emptyset$? Let us check

$$\sum_{i\in\emptyset} x_i \geq v_{L,R}(\emptyset).$$

Since there is not i from $\emptyset$ (otherwise $\emptyset$ would not be the empty set), the sum $\sum_{i\in\emptyset} x_i$ has no summands and is equal to zero. Since all coalition functions have worth zero for the empty set, we find $\sum_{i\in\emptyset} x_i = 0 = v_{L,R}(\emptyset)$ for the gloves game and also for any coalition function.

EXERCISE III.6. *Determine the core for the gloves game $v_{L,R}$ with $L = \{1, 2\}$ and $R = \{3\}$.*

In case of $|L| = 2 > 1 = |R|$ right gloves are scarcer than left gloves. In such a situation, the owner of a right glove should be better off than the owner of a left glove. The core reflects the relative scarcity in a drastic way. Consider the Pareto-efficient payoff vector

$$y = \left(\frac{1}{10}, \frac{1}{10}, \frac{8}{10}\right)$$

that does not lie in the core. This payoff vector can be blocked by coalition $\{1, 3\}$. Its worth is $v(\{1, 3\}) = 1$ which can be distributed among its members in a manner that both are better off.

Note that the core is a set-valued solution concept. It can contain one payoff vector (see the above exercise) or very many payoff vectors (in case of $L = \{1\}$ and $R = \{2\}$). Later on, we will see coalition functions with an empty core: every feasible payoff vector is blockable by at least one coalition.

The core for coalition functions has first been defined by Gillies (1959). Shubik (1981, S. 299) mentions that Lloyd Shapley proposes this concept as early as 1953 in unpublished lecture notes. In the framework of an exchange economy, Edgeworth (1881) proposes a very similar concept (see chapter **??**).

6. The Shapley value: the formula

In contrast to Pareto efficiency and the core, the Shapley value is a point-valued solution concept, i.e., a solution function. For every coalition function, it spits out exactly one payoff vector. Shapley's (1953) article is famous for pioneering the twofold approach of algorithm and axioms.

We begin with the Shapley formula. It rests on a simple idea. Every player obtains

- an average of
- his marginal contributions.

Beginning with the latter, the marginal contribution of player i with respect to coalition K is

"the value with him" minus "the value without him".

Thus, the marginal contributions reflect a player's productivity:

Definition III.6 (marginal contribution). *Let $i \in N$ be a player from N and let $v \in V(N)$ be a coalition function on N. Player i's marginal contribution with respect to a coalition K is denoted by $MC_i^K(v)$ and given by*

$$MC_i^K(v) := v(K \cup \{i\}) - v(K \backslash \{i\}).$$

The marginal contribution of a player depends on the coalition function and the coalition. It does not matter whether i is a member of K or not, i.e., we have $MC_i^{K\cup\{i\}}(v) = MC_i^{K\backslash\{i\}}(v)$.

Exercise III.7. *Determine the marginal contributions for $v_{\{1,2,3\},\{4,5\}}$ and*

- $i = 1, K = \{1, 3, 4\}$,
- $i = 1, K = \{3, 4\}$,
- $i = 4, K = \{1, 3, 4\}$,
- $i = 4, K = \{1, 3\}$.

We now need to explain the kind of averaging employed by the Shapley formula. In order to calculate the Shapley value, one considers all rank orders of the n players. $(3, 1, 2)$ is one rank order of the players 1 to 3. Just imagine that the players 3, 1 and 2 stand out side the door and enter, one after the other. We are interested in the marginal contributions. For rank order $(3, 1, 2)$, one finds the marginal contributions

$$v(\{3\}) - v(\emptyset), v(\{1,3\}) - v(\{3\}) \text{ and } v(\{1,2,3\}) - v(\{1,3\}).$$

They add up to $v(N) - v(\emptyset) = v(N)$.

Definition III.7 (rank order). *Let $N = \{1, ..., n\}$ be a player set. Bijective function $\rho : N \to N$ are called rank orders or permutations on N. The set of all permutations on N is denoted by RO_N. The set of all players "up to and including player i under rank order ρ" is denoted by $K_i(\rho)$ and given by*

$$\rho(j) = i \text{ and } K_i(\rho) = \{\rho(1), .., \rho(j)\}.$$

Player i's marginal contribution with respect to rank order K is denoted by $MC_i^\rho(v)$ and given by

$$MC_i^\rho(v) := MC_i^{K_i(\rho)}(v) = v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\}).$$

EXERCISE III.8. *Find player 2's marginal contributions for the rank orders $(1,3,2)$ and $(3,1,2)$!*

For every player, his Shapley value is the average of his marginal contributions where each rank order is equally likely. Thus, we can employ the following algorithm:

- We first determine all the possible rank orders.
- We then find the marginal contributions for every rank order.
- For every player, we add his marginal contributions.
- Finally, we divide the sum by the number of rank orders.

Consider the simple example given by $N = \{1,2,3\}$, $L = \{1,2\}$ and $R = \{3\}$. We find the rank orders:

$$\begin{gathered}(1,2,3),(1,3,2),\\(2,1,3),(2,3,1),\\(3,1,2),(3,2,1).\end{gathered}$$

For three players, there are $1 \cdot 2 \cdot 3 = 6$ different rank orders. It is not difficult to see, why. For a single player 1, we have just one rank order (1). The second player 2 can be placed before or after player 1 so that we obtain the $1 \cdot 2$ rank orders

$$\begin{gathered}(1,2),\\(2,1).\end{gathered}$$

For each of these two, the third player 2 can be placed before the two players, in between or after them:

$$\begin{gathered}(3,1,2),(1,3,2),(1,2,3),\\(3,2,1),(2,3,1),(2,1,3).\end{gathered}$$

Therefore, we have $2 \cdot 3 = 6$ rank orders. Generalizing, , for n players, we have $1 \cdot 2 \cdot ... \cdot n$ rank orders. We can also use the abbreviation

$$n! := 1 \cdot 2 \cdot ... \cdot n$$

which is to be read "n factorial".

EXERCISE III.9. *Determine the number of rank oders for 5 and for 6 players!*

EXERCISE III.10. *Consider $N = \{1,2,3\}$, $L = \{1,2\}$ and $R = \{3\}$ and determine player 1's marginal contribution for each rank order.*

DEFINITION III.8 (Shapley value). *The Shapley value is the solution function Sh given by*

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^\rho(v)$$

According to the previous exercise, we have

$$Sh_1\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}.$$

The Shapley values of the other two players can be obtained by the same procedure. However, there is a more elegant possibility. The Shapley values of players 1 and 2 are identical because they hold a left glove each and are symmetric (in a sense to be defined shortly). Thus, we have $Sh_2\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}$. Also, the Shapley value satisfies Pareto efficiency which means that the sum of the payoffs equals the worth of the grand coalition:

$$\sum_{i=1}^{3} Sh_i\left(v_{\{1,2\},\{3\}}\right) = v(\{1,2,3\}) = 1$$

Thus, we find

$$Sh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right).$$

7. The Shapley value: the axioms

The Shapley value fulfills four axioms:

- the efficiency axiom: the worth of the grand coalition is to be distributed among all the players,
- the symmetry axiom: players in similar situations obtain the same payoff,
- the null-player axiom: a player with zero marginal contribution to every coalition, obtains zero payoff, and
- additivity axiom: if players are subject to two coalition functions, it does not matter whether we apply the Shapley value to the sum of these two coalition functions or apply the Shapley value to each coalition function separately and sum the payoffs.

A solution function σ may or may not obey the four axioms mentioned above.

DEFINITION III.9 (efficiency axiom). *A solution function σ is said to obey the efficiency axiom or the Pareto axiom if*

$$\sum_{i \in N} \sigma_i(v) = v(N)$$

holds for all coalition functions $v \in V$.

In the gloves game, two left-glove owners are called symmetric.

DEFINITION III.10 (symmetry). *Two players i and j are called symmetric (with respect to $v \in V$) if we have*

$$v(K \cup \{i\}) = v(K \cup \{j\})$$

for every coalition K that does not contain i or j.

EXERCISE III.11. *Show that any to left-glove holders are symmetric in a gloves game $v_{L,R}$.*

EXERCISE III.12. *Show $MC_i^K = MC_j^K$ for two symmetric players i and j fulfilling $i \notin K$ and $j \notin K$.*

It may seem obvious that symmetric players obtain the same payoff:

DEFINITION III.11 (symmetry axiom). *A solution function σ is said to obey the symmetry axiom if we have*

$$\sigma_i(v) = \sigma_j(v)$$

for any game $v \in V$ and any two symmetric players i and j.

In any gloves game obeying $L \neq \emptyset \neq R$, every player has a non-zero marginal contribution sometimes.

DEFINITION III.12 (null player). *A player $i \in N$ is called a null player (with respect to v) if*

$$v(K \cup \{i\}) = v(K)$$

holds for every coalition K.

Shouldn't a null player obtain nothing?

DEFINITION III.13 (null-player axiom). *A solution function σ is said to obey the null-player axiom if we have*

$$\sigma_i(v) = 0$$

for any game $v \in V$ and for any null player $i \in N$.

EXERCISE III.13. *Under which condition is a player from L a null player in a gloves game $v_{L,R}$?*

The last axiom that we consider at present is the additivity axiom. It rests on the possibility to add both payoff vectors and coalition functions (see section 2).

DEFINITION III.14 (additivity axiom). *A solution function σ is said to obey the additivity axiom if we have*

$$\sigma(v + w) = \sigma(v) + \sigma(w)$$

for any two coalition functions $v, w \in V$ with $N(v) = N(w)$.

Do you see the difference? On the left-hand side, we add the coalition functions first and then apply the solution function. On the right-hand side we apply the solution function to the coalition functions individually and then add the payoff vectors.

EXERCISE III.14. *Can you deduce* $\sigma(0) = 0$ *from the additivity axiom? Hint: use* $v = w := 0$.

Now we note a stunning result:

THEOREM III.1 (Shapley axiomatization). *The Shapley formula is the unique solution function that fulfills the symmetry axiom, the efficiency axiom, the null-player axiom and the additivity axiom.*

The theorem means that the Shapley formula fulfills the four axioms. Consider now a solution function that fulfills the four axioms. According to the theorem, the Shapley formula is the only solution function to do so.

Differently put, the Shapley formula and the four axioms are equivalent – they specify the same payoffs. Cooperative game theorists say that she Shapley formula is "axiomatized" by the set of the four axioms. The chapter after next will show you how to prove this wonderful result.

EXERCISE III.15. *Determine the Shapley value for the gloves game for* $L = \{1\}$ *and* $R = \{2, 3, 4\}$! *Hint: You do not need to write down all* 4! *rank orders. Try to find the probability that player* 1 *does not complete a pair.*

8. Topics and literature

The main topics in this chapter are

- coalition
- coalition function
- gloves game
- core
- efficiency
- feasibility
- marginal contribution
- axioms
- symmetry
- null player
- Shapley value

We introduce the following mathematical concepts and theorems:

- t

-

We recommend the textbook by Wiese (2005).

9. Solutions

Exercise III.1

The values are

$$\begin{aligned} v_{L,R}(\{1\}) &= \min(1,0) = 0, \\ v_{L,R}(\emptyset) &= \min(0,0) = 0, \\ v_{L,R}(N) &= \min(2,3) = 2 \text{ and} \\ v_{L,R}(\{2,3,4\}) &= \min(2,1) = 1. \end{aligned}$$

Exercise III.2

The first three propositions are nonsensical, the last one is correct.

Exercise III.3

We obtain the sum of vectors

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 3+5 \\ 6+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$$

Exercise III.4

Feasibility follows from

$$\begin{aligned}\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{n}\left(v_{L,R}(N) - \sum_{j=1}^{n} x_j\right)\\ &= \sum_{i=1}^{n} x_i + \frac{1}{n}\left(\sum_{i=1}^{n} v_{L,R}(N) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_j\right)\\ &= \sum_{i=1}^{n} x_i + \frac{1}{n}\left(n v_{L,R}(N) - n\sum_{j=1}^{n} x_j\right)\\ &= v_{L,R}(N).\end{aligned}$$

Exercise III.5

The set of Pareto-efficient payoff vectors (x_1, x_2) are described by $x_1 + x_2 = 1$. In particular, we may well have $x_1 < 0$.

Exercise III.6

The core obeys the conditions

$$\begin{aligned}x_1 + x_2 + x_3 &= v_{L,R}(N) = 1,\\ x_i &\geq 0, i = 1, 2, 3,\\ x_1 + x_2 &\geq 0,\\ x_1 + x_3 &\geq 1 \text{ and}\\ x_2 + x_3 &\geq 1.\end{aligned}$$

Substituting $x_1 + x_3 \geq 1$ into the efficiency condition yields

$$x_2 = 1 - (x_1 + x_3) \leq 1 - 1 = 0.$$

Hence (because of $x_2 \geq 0$), we have $x_2 = 0$. For reasons of symmetry, we also have $x_1 = 0$. Applying efficiency once again, we obtain $x_3 = 1 - (x_1 + x_2) = 1$. Thus, the only candidate for the core is $x = (0, 0, 1)$. Indeed, this payoff vector fulfills all the conditions noted above. Therefore,

$$(0, 0, 1)$$

is the only element in the core.

Exercise III.7

The marginal contributions are

$$\begin{aligned}
MC_1^{\{1,3,4\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v\left(\{1,3,4\}\cup\{1\}\right)-v\left(\{1,3,4\}\setminus\{1\}\right)\\
&= v\left(\{1,3,4\}\right)-v\left(\{3,4\}\right)\\
&= 1-1=0,\\
MC_1^{\{3,4\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v\left(\{3,4\}\cup\{1\}\right)-v\left(\{3,4\}\setminus\{1\}\right)\\
&= v\left(\{1,3,4\}\right)-v\left(\{3,4\}\right)\\
&= 1-1=0,\\
MC_4^{\{1,3,4\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v\left(\{1,3,4\}\cup\{4\}\right)-v\left(\{1,3,4\}\setminus\{4\}\right)\\
&= v\left(\{1,3,4\}\right)-v\left(\{1,3\}\right)\\
&= 1-0=1,\\
MC_4^{\{1,3\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v\left(\{1,3\}\cup\{4\}\right)-v\left(\{1,3\}\setminus\{4\}\right)\\
&= v\left(\{1,3,4\}\right)-v\left(\{1,3\}\right)\\
&= 1-0=1.
\end{aligned}$$

Exercise III.8

The marginal contributions are the same: $v\left(\{1,2,3\}\right)-v\left(\{1,3\}\right)$.

Exercise III.9

We find $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$ rank orders of 5 players and $6! = 5!\cdot 6 = 120\cdot 6 = 720$ rank orders for 6 players.

Exercise III.10

We find the marginal contributions

$$\begin{aligned}
v\left(\{1\}\right)-v\left(\emptyset\right) &= 0-0=0,\text{ rank order }(1,2,3)\\
v\left(\{1\}\right)-v\left(\emptyset\right) &= 0-0=0,\text{ rank order }(1,3,2)\\
v\left(\{1,2\}\right)-v\left(\{2\}\right) &= 0-0=0,\text{ rank order }(2,1,3)\\
v\left(\{1,2,3\}\right)-v\left(\{2,3\}\right) &= 1-1=0,\text{ rank order }(2,3,1)\\
v\left(\{1,3\}\right)-v\left(\{3\}\right) &= 1-0=1,\text{ rank order }(3,1,2)\\
v\left(\{1,2,3\}\right)-v\left(\{2,3\}\right) &= 1-1=0,\text{ rank order }(3,2,1).
\end{aligned}$$

Exercise III.11

Let i and j be players from L and let K be a coalition that contains neither i nor j. Then $K\cup\{i\}$ contains the same number of left and the same number of right gloves as $K\cup\{j\}$. Therefore,

$$\begin{aligned}
v_{L,R}\left(K\cup\{i\}\right) &= \min\left(\left|(K\cup\{i\})\cap L\right|,\left|(K\cup\{i\})\cap R\right|\right)\\
&= \min\left(\left|(K\cup\{j\})\cap L\right|,\left|(K\cup\{j\})\cap R\right|\right)\\
&= v_{L,R}\left(K\cup\{j\}\right).
\end{aligned}$$

Exercise III.12

The equality follows from

$$\begin{aligned} MC_i^K &= v(K \cup \{i\}) - v(K \backslash \{i\}) \\ &= v(K \cup \{i\}) - v(K) \\ &= v(K \cup \{j\}) - v(K) \\ &= v(K \cup \{j\}) - v(K \backslash \{j\}) \\ &= MC_j^K. \end{aligned}$$

Exercise III.13

A player i from L is a null player iff $R = \emptyset$ holds. $R = \emptyset$ implies

$$\begin{aligned} v_{L,\emptyset}(K) &= \min(|K \cap L|, |K \cap \emptyset|) \\ &= \min(|K \cap L|, 0) \\ &= 0 \end{aligned}$$

for every coalition K. $R \neq \emptyset$ means that i has a marginal contribution of 1 when he comes second after a right-glove holder.

Exercise III.15

The left-glove holder 1 completes a pair (the only one) whenever he does not come first. The probability for coming first is $\frac{1}{4}$ for player 1 (and any other player). Thus, player 1 obtains $\left(1 - \frac{1}{4}\right) \cdot 1$. The other players share the rest. Therefore, symmetry and efficiency lead to

$$\begin{aligned} \varphi_1\left(v_{\{1\},\{2,3,4\}}\right) &= \frac{3}{4}, \\ \varphi_2\left(v_{\{1\},\{2,3,4\}}\right) &= \varphi_3\left(v_{\{1\},\{2,3,4\}}\right) = \varphi_4\left(v_{\{1\},\{2,3,4\}}\right) = \frac{1}{12}. \end{aligned}$$

10. Further exercises without solutions

CHAPTER IV

Many games

In the previous chapter, we focus on a specific class of games, the gloves games. In this chapter, we aim to familiarize the reader with many other interesting games.

1. Simple games

1.1. Definition. We first define monotonic games and then simple games.

Definition IV.1 (monotonic game). *A coalition function* $v \in V(N)$ *is called monotonic if* $\emptyset \subseteq S \subseteq S'$ *implies* $v(S) \leq v(S')$.

Thus, monotonicity means that the worth of a coalition cannot decrease if other players join. Simple games are a special subclass of monotonic games:

Definition IV.2 (simple game). *A coalition function* $v \in V(N)$ *is called simple if*

- *we have* $v(K) = 0$ *or* $v(K) = 1$ *for every coalition* $K \subseteq N$,
- *the grand coalition's worth is* 1 *and.*
- v *is monotonic.*

Thus, if S' is a superset of S (or S a subset of S'), we cannot have $v(S) = 1$ and $v(S') = 0$. Since the worths are 0 or 1, we can distinguish groups of agents accordingly:

Definition IV.3 (winning coalition, loosing coalition). *Let* v *be a simple coalition function. Coalitions with* $v(K) = 1$ *are called winning coalitions and coalitions with* $v(K) = 0$ *are called loosing coalitions. A winning coalition* K *is a minimal winning coalition if every strict subset of* K *is not a winning coalition.*

Simple games can be characterized by the pivotal coalitions of all the players:

Definition IV.4 (pivotal coalition). *For a simple game* v, $K \subseteq N$ *is a pivotal coalition for* $i \in N$ *if* $v(K) = 0$ *and* $v(K \cup \{i\}) = 1$. *The number of* i*'s pivotal coalitions is denoted by* $\eta_i(v)$,

$$\eta_i(v) := |\{K \subseteq N : v(K) = 0 \text{ and } v(K \cup \{i\}) = 1\}|.$$

We have $\eta(v) := (\eta_1(v), ..., \eta_n(v))$ *and* $\bar{\eta}(v) := \sum_{i\in N} \eta_i(v)$. *We sometimes omit the game and write* η_i $(\eta, \bar{\eta})$ *rather than* $\eta_i(v)$ $(\eta(v), \bar{\eta}(v))$.

By $\left|2^{N\backslash\{i\}}\right| = 2^{n-1}$, no player can have more pivotal coalitions than 2^{n-1}.

Exercise IV.1. *How do you call a player* $i \in N$ *who has no pivotal coalitions? How is player* i *called who fulfills* $\eta_i = 2^{n-1}$*?*

1.2. Veto players and dictators. According to the previous exercise, all interesting simple games have $v(N) = 1$. Sometimes, some players are of central importance:

Definition IV.5 (veto player, dictator). *Let* v *be a simple game. A player* $\in N$ *is called a veto player if*

$$v(N\backslash\{i\}) = 0$$

holds. i *is called a dictator if*

$$v(S) = \begin{cases} 1, & i \in S \\ 0, & \textit{sonst} \end{cases}$$

holds for all $S \subseteq N$.

Thus, without a veto player, the worth of a coalition is 0 while a dictator can produce the worth 1 just by himself.

Exercise IV.2. *Can there be a coalition* K *such that* $v(K\backslash\{i\}) = 1$ *for a veto player* i *or a dictator* i?

Exercise IV.3. *Is every veto player a dictator or every dictator a veto player?*

1.3. Simple games and voting mechanisms. Oftentimes, simple games can often be used to model voting mechanisms. We need the set-theoretic concept of a complement:

Definition IV.6 (complement). *The set* $N\backslash K := \{i \in N : i \notin K\}$ *is called* K*'s complement (with respect to* N*).*

As a matter of consistency, complements of winning coalitions have to be loosing coalitions. Otherwise, a coalition K could vote for something and $N\backslash K$ would vote against it, both of them successfully.

Definition IV.7 (contradictory, decidable). *A simple game* $v \in V(N)$ *is called non-contradictory if* $v(K) = 1$ *implies* $v(N\backslash K) = 0$.

A simple game $v \in V(N)$ *is called decidable if* $v(K) = 0$ *implies* $v(N\backslash K) = 1$.

Thus, a contradictory voting game can lead to opposing decisions – for example, some candidate A is voted president (with the support of some coalition K) and then some other candidate B (with the support of $N\backslash K$) is also voted president. A non-decidable voting game can prevent any decision. Neither A nor B can gain enough support because coalition K blocks candidate B while $N\backslash K$ blocks candidate A.

EXERCISE IV.4. *Show that a simple game with a veto player cannot be contradictory. A simple game with two veto players cannot be decidable.*

1.4. Unanimity games. Unanimity games are famous games in cooperative game theory. We will use them to prove the Shapley theorem.

DEFINITION IV.8 (unanimity game). *For any $T \neq \emptyset$,*

$$u_T(K) = \begin{cases} 1, & K \supseteq T \\ 0, & \textit{otherwise} \end{cases}$$

defines a unanimity game.

We can interpret the players from T as the productive or powerful members of society. They generate the worth of 1. For example, each player $i \in T$ possesses part of a treasure map. The treasure can be found only if all the different parts of the map are put together.

Every player from T is a veto player and no player from $N\backslash T$ is a veto player. In a sense, the players from T exert common dictatorship.

EXERCISE IV.5. *Find the null players in the unanimity game u_T.*

EXERCISE IV.6. *Find the core and the Shapley value for $N = \{1, 2, 3, 4\}$ and $u_{\{1,2\}}$.*

1.5. Apex-Spiel. The apex game has one important player $i \in N$ who is nearly a veto player and nearly a dictator.

DEFINITION IV.9 (apex game). *For $i \in N$ with $n \geq 2$, the apex game h_i is defined by*

$$h_i(K) = \begin{cases} 1, & i \in K \textit{ and } K\backslash\{i\} \neq \emptyset \\ 1, & K = N\backslash\{i\} \\ 0, & \textit{otherwise} \end{cases}$$

Player i is called the main, or apex, player of that game.

Thus, there are two types of winning coalitions in the apex game:

- i together with at least one other player or
- all the other players taken together.

Generally, we work with apex games for $n \geq 4$.

EXERCISE IV.7. *Consider h_1 for $n = 2$ and $n = 3$. How do these games look like?*

EXERCISE IV.8. *Is the apex player a veto player or a dictator?*

EXERCISE IV.9. *Show that the apex game is not contradictory and decidable.*

Let us now think find the Shapley value for the apex game. Consider all the rank orders. The apex player $i \in N$ obtains the marginal contribution 1 unless

- he is the first player in a rank order (then his marginal contribution is $v(\{i\}) - v(\emptyset) = 0 - 0 = 0$) or
- he is the last player (with marginal contribution $v(N) - v(N\backslash\{i\}) = 1 - 1 = 0$).

Since every position of the apex player in a rank order has the same probability, the following exercise is easy:

EXERCISE IV.10. *Find the Shapley value for the apex game h_1!*

1.6. Weighted voting games.

1.6.1. *Definition.* Weighted voting games form an important subclass of the simple games. We specify weights for every player and a quota. If the sum of weights for a coalition is equal to or above the quota, that coalition is a winning one.

DEFINITION IV.10 (weighted voting game). *A voting game v is specified by a quota q and voting weights g_i, $i \in N$, and defined by*

$$v(K) = \begin{cases} 1, & \sum_{i \in K} g_i \geq q \\ 0, & \sum_{i \in K} g_i < q \end{cases}$$

In that case, the voting game is also denoted by $[q; g_1, ..., g_n]$.

For example,

$$\left[\frac{1}{2}; \frac{1}{n}, ..., \frac{1}{n}\right]$$

is the majority rule, according to which fifty percent of the votes are necessary for a winning coalition. Do you see that $n = 4$ implies that the coalition $\{1, 2\}$ is a winning coalition and also the coalition of the other players, $\{3, 4\}$? Thus, this voting game is contradictory.

The apex game h_1 for n players can be considered a weighted voting game given by

$$\left[n - 1; n - \frac{3}{2}, 1, ..., 1\right].$$

EXERCISE IV.11. *Consider the unanimity game u_T given by $t < n$ and $T = \{1, ..., t\}$. Can you express it as a weighted voting game?*

1.6.2. *UN Security Council.* Let us consider the United Nations' Security Council. According to http://www.un.org/sc/members.asp:, it has 5 permanent members and 10 non-permanent ones. The permanent members are China, France, Russian Federation, the United Kingdom and the United States. In 2009, the non-permanent members were Austria, Burkina Faso, Costa Rica, Croatia, Japan, Libyan Arab Jamahiriya, Mexico, Turkey, Uganda and Viet Nam.

We read:

> Each Council member has one vote. ... Decisions on substantive matters require nine votes, including the concurring votes of all five permanent members. This is the rule of "great Power unanimity", often referred to as the "veto" power.
>
> Under the Charter, all Members of the United Nations agree to accept and carry out the decisions of the Security Council. While other organs of the United Nations make recommendations to Governments, the Council alone has the power to take decisions which Member States are obligated under the Charter to carry out.

Obviously, the UN Security Council has a lot of power and so its voting mechanism deserves analysis. The above rule for "substantive matters" can be translated into the weighted voting game

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

where the weights 7 accrue to the five permanent and the weights 1 to the non-permanent members.

EXERCISE IV.12. *Using the above voting game* $[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$, *show that every permanent member is a veto player. Show also that the five permanent members need the additional support of four non-permanent ones.*

EXERCISE IV.13. *Is the Security Council's voting rule non-contradictory and decidable?*

It is not easy to calculate the Shapley value for the Security Council. After all, we have

$$15! = 1.307.674.368.000$$

rank orders for the 15 players. Anyway, the Shapley values are

$$0,19627 \text{ for each permanent member}$$
$$0,00186 \text{ für each non-permanent member.}$$

2. Three non-simple games

2.1. Buying a car. Following Morris (1994, S. 162), we consider three agents envolved in a car deal. Andreas (A) has a used car he wants to sell, Frank (F) and Tobias (T) are potential buyers with willingness to buy of 700 and 500, respectively. This leads to the coalition function v given by

$$\begin{aligned} v(A) &= v(F) = v(T) = 0, \\ v(A,F) &= 700, \\ v(A,T) &= 500, \\ v(F,T) &= 0 \text{ and} \\ v(A,F,T) &= 700. \end{aligned}$$

One-man coalitions have the worth zero. For Andreas, the car is useless (he believes in cycling rather than driving). Frank and Tobias cannot obtain the car unless Andreas cooperates. In case of a deal, the worth is equal to the (maximal) willingness to pay.

We use the core to find predictions for the car price. The core is the set of those payoff vectors (x_A, x_F, x_T) that fulfill

$$x_A + x_F + x_T = 700$$

and

$$\begin{aligned} x_A &\geq 0, x_F \geq 0, x_T \geq 0, \\ x_A + x_F &\geq 700, \\ x_A + x_T &\geq 500 \text{ and} \\ x_F + x_T &\geq 0. \end{aligned}$$

Tobias obtains

$$\begin{aligned} x_T &= 700 - (x_A + x_F) \text{ (efficiency)} \\ &\leq 700 - 700 \text{ (by } x_A + x_F \geq 700) \\ &= 0 \end{aligned}$$

and hence zero, $x_T = 0$. By $x_A + x_T \geq 500$, the seller Andreas can obtain at least 500.

Summarizing (and checking all the conditions above), we see that the core is the set of vectors (x_A, x_F, x_T) obeying

$$\begin{aligned} 500 &\leq x_A \leq 700, \\ x_F &= 700 - x_A \text{ and} \\ x_T &= 0. \end{aligned}$$

Therefore, the car sells for a price between 500 and 700.

2.2. The Maschler game. Aumann & Myerson (1988) present the Maschler game which is the three-player game given by

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

Obviously, the three players are symmetric. It is easy to see that all players of symmetric games are symmetric.

Definition IV.11 (symmetric game). *A coalition function v is called symmetric if there is a function $f : N \to \mathbb{R}$ such that*

$$v(K) = f(|K|), \ K \subseteq N.$$

Exercise IV.14. *Find the Shapley value for the Maschler game!*

According to the Shapley value, the players 1 and 2 obtain less than their common worth. Therefore, they can block the payoff vector suggested by the Shapley value. Indeed, for any efficient payoff vector, we can find a two-man coalition that can be made better off. Differently put: the core is empty.

This can be seen easily. We are looking for vectors (x_1, x_2, x_3) that fulfill both

$$x_1 + x_2 + x_3 = 72$$

and

$$\begin{aligned} x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0, \\ x_1 + x_2 &\geq 60, \\ x_1 + x_3 &\geq 60 \text{ and} \\ x_2 + x_3 &\geq 60. \end{aligned}$$

Summing the last three inequalities yields

$$2x_1 + 2x_2 + 2x_3 \geq 3 \cdot 60 = 180$$

and hence a contradiction to efficiency.

2.3. The gloves game, once again. In chapter III, we have calculated the core for the gloves game $L = \{1, 2\}$ and $R = \{3\}$. The core clearly shows the bargaining power of the right-glove owner. We will now consider the core for a case where the scarcity of right gloves seems minimal:

$$\begin{aligned} L &= \{1, 2, ..., 100\} \\ R &= \{101, ..., 199\}. \end{aligned}$$

If a payoff vector

$$(x_1, ..., x_{100}, x_{101}, ..., x_{199})$$

is to be long to the core, we have

$$\sum_{i=1}^{199} x_i = 99$$

by the efficiency axiom. We now pick any left-glove holder $j \in \{1, 2, ..., 100\}$. We find

$$v\left(L \backslash \{j\} \cup R\right) = 99$$

and hence

$$\begin{aligned} x_j &= 99 - \sum_{\substack{i=1, \\ i \neq j}}^{199} x_i \text{ (efficiency)} \\ &\leq 99 - 99 \text{ (blockade by coalition } L \backslash \{j\} \cup R) \\ &= 0. \end{aligned}$$

Therefore, we have $x_j = 0$ for every $j \in L$.

Every right-glove owner can claim at least 1 because he can point to coalitions where he is joined by at least one left-glove owner. Therefore, every right-glove owner obtains the payoff 1 and every left-glove owner the payoff zero. Inspite of the minimal scarcity, the right-glove owners get everything.

If two left-glove owners burned their glove, the other left-glove owners would get a payoff increase from 0 to 1. (Why?)

Exercise IV.15. *Consider a generalized gloves game where*

- *player* 1 *has one left glove,*
- *player* 2 *has two left gloves and*
- *players* 3 *and* 4 *have one right glove each.*

Calculate the core. How does the core change if player 2 *burns one of his two gloves?*

The burn-a-glove strategy may make sense if payoffs depend on the scarcity in an extreme fashion as they do for the core.

3. Cost division games

Many organizations have the problem of dividing overhead cost to several units. Examples are doctors with a common secretary or commonly used facilities, firms organized as a collection of profit-centers, universities with computing facilities used by several departments or faculties. This chapter rests on Young (1994a) and chapter 5 from Young (1994b).

Definition IV.12 (cost-division game). *For a player set N, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of c,* *the cost-savings game is defined by* $v : 2^N \to \mathbb{R}$ *and*

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K), K \subseteq N.$$

The idea behind this definition is that cost savings can be realized if players pool their resources so that $\sum_{i \in K} c(\{i\})$ is greater than $c(K)$ and $v(K)$ is positive.

We consider a specific example. Two towns A and B plan a water-distribution system.Town A could build such a system for itself at a cost of 11 million Euro and twon B would need 7 million Euro for a system tailor-made to its needs. The cost for a common water-distribution system is 15 million Euro. The cost function is given by

$$\begin{aligned} c(\{A\}) &= 11, c(\{B\}) = 7 \text{ and} \\ c(\{A,B\}) &= 15. \end{aligned}$$

The associated cost-savings game is $v : 2^{\{A,B\}} \to \mathbb{R}$ defined by

$$\begin{aligned} v(\{A\}) &= 0, c(\{B\}) = 0 \text{ and} \\ v(\{A,B\}) &= 7 + 11 - 15 = 3. \end{aligned}$$

v's core is obviously given by

$$\left\{(x_A, x_B) \in \mathbb{R}^2_+ : x_1 + x_2 = 3\right\}.$$

The cost savings of $3 = 11 + 7 - 15$ can be allotted to the towns such that no town is worse off compared to going alone. Thus, the set of undominated cost allocations is

$$\left\{(c_A, c_B) \in \mathbb{R}^2 : c_A + c_B = 15, c_A \leq 11, c_B \leq 7\right\}.$$

4. Endowment games

Gloves games are a specific class of endowment games. In these games, players own an endowment (in the gloves game: a right or a left glove). We first define the endowment economy and then, on that basis, the endowment game.

Definition IV.13 (endowment economy). *An endowment economy is a tuple*

$$\mathcal{E} = \left(N, G, \left(\omega^i\right)_{i \in N}, agg\right)$$

consisting of

- *the set of agents* $N = \{1, 2, ..., n\}$,
- *the finite set of goods* $G = \{1, ..., \ell\}$,
- *for every agent* $i \in N$, *an endowment* $\omega^i = \left(\omega^i_1, ..., \omega^i_\ell\right) \in \mathbb{R}^\ell_+$ *where*

$$\omega := \sum_{i \in N} \omega^i = \left(\sum_{i \in N} \omega^i_1, ..., \sum_{i \in N} \omega^i_\ell\right)$$

is the economy's total endowment, and
- *an aggregation functions* $agg : \mathbb{R}^\ell \to \mathbb{R}$.

The aggregation function aggregates the different goods' amounts into a specific real number in the same way as the min-operator does in the gloves game.

Definition IV.14 (endowment game). *Consider an endowment economy* $\mathcal{E}$. *An endowment game* $v^{\mathcal{E}} : 2^N \to \mathbb{R}$ *is defined by*

$$v^{\mathcal{E}}(K) := agg\left(\sum_{i\in K}\omega_1^i, ..., \sum_{i\in K}\omega_\ell^i\right).$$

Within the class of endowment games, we can define the sum of two coalition functions on N in the usual manner – just sum the worths of every coalition. For example, we have

$$\begin{aligned} & \left(v_{\{1,2\},\{3\}} + v_{\{1\},\{2,3\}}\right)(\{2\}) \\ = \; & v_{\{1,2\},\{3\}}(\{2\}) + v_{\{1\},\{2,3\}}(\{2\}) \\ = \; & 0 + 0 = 0 \end{aligned}$$

However, taking the specific nature of endowment games into account, it is also plausible to sum endowments and take it from there. In that case, we find that player 2 has a left glove (in $v_{\{1,2\},\{3\}}$) and a right glove (in $v_{\{1\},\{2,3\}}$) and hence the worth 1. We capture this idea by the following definition:

Definition IV.15 (summing of endowment games). *Consider two endowment economies* $\mathcal{E}$ *and* $\mathcal{F}$ *which have the same player set* N*, the same set of goods* G *and the same aggregation function* agg*. In that case,* $\mathcal{E}$ *and* $\mathcal{F}$ *are called structurally identical. The (possibly different) endowments are denoted* $\omega_{\mathcal{E}}$ *and* $\omega_{\mathcal{F}}$*, respectively, and the derived endowment games by* $v_{\mathcal{E}}$ *and* $v_{\mathcal{F}}$*. The endowment-based sum of these games is denoted by* $v_{\mathcal{E}} \oplus v_{\mathcal{F}}$ *and defined by*

$$\begin{aligned} \omega_g^i &= (\omega_{\mathcal{E}})_g^i + (\omega_{\mathcal{F}})_g^i, \; i \in N, g \in G \text{ and} \\ (v_{\mathcal{E}} \oplus v_{\mathcal{F}})(K) &:= agg\left(\sum_{i\in K}\omega_1^i, ..., \sum_{i\in K}\omega_\ell^i\right). \end{aligned}$$

Note that the sum of two gloves games need not be a gloves game, but a generalized gloves game where players can have any number of left or right gloves.

Endowment-based summing is of economic interest. For example, we can consider two autarkic economies that open up for trade and define the gains from trade:

Definition IV.16 (summing of endowment games). *For a player set* N*, consider two endowment economies* $\mathcal{E}$ *and* $\mathcal{F}$ *. The gains from trade are* *defined by*

$$GfT(\mathcal{E},\mathcal{F}) = (v_{\mathcal{E}} \oplus v_{\mathcal{F}})(N) - [v_{\mathcal{E}}(N) + v_{\mathcal{F}}(N)].$$

Thus the usual sum of coalition function ignores all substantial linkages that might exist between them.

EXERCISE IV.16. *Show that the gains from trade are zero for any gloves game* $v_{\mathcal{E}} := v_{\{L\},\{R\}}$ *and* $v_{\mathcal{F}} := v_{\mathcal{E}}$.

A specific class of endowment games has been proposed by Owen (1975): production games. In these games, players' endowments represent factors of production rather than consumption goods. The idea is that the players pool their factors of production and sell the output. We define the aggregation function $agg : \mathbb{R}^{\ell} \to \mathbb{R}$ by

$$agg(\omega_1, ..., \omega_\ell) := p \cdot f(\omega_1, ..., \omega_\ell)$$

where f is a production function and p the price vector. If m goods are produced, p is a price vector with m entries and $\cdot$ stands for the scalar product. Thus, the endowment game's worths stand for

- the revenue
- generated by the output
- produced with the factors of production
- a coalition is endowed with.

5. Properties of coalition functions

5.1. Zero players and symmetric players.

DEFINITION IV.17 (zero player). *A player* $i \in N$ *is a zero player for a coalition function* $v \in V(N)$ *if*

$$v(K \cup \{i\}) = v(K \backslash \{i\})$$

holds for every coalition $K \subseteq N$.

DEFINITION IV.18 (inessential player). *A player* $i \in N$ *is an inessential player for a coalition function* $v \in V(N)$ *if*

$$v(K \cup \{i\}) - v(K \backslash \{i\}) = v(\{i\})$$

holds for every coalition $K \subseteq N$.

5.2. Inessentiality and additivity. We begin with boring coalition functions.

DEFINITION IV.19 (triviality). *A coalition function* $v \in V(N)$ *is called trivial if*

$$v(K) = 0$$

holds for every coalition $K \subseteq N$.

Thus, a trivial coalition function $v \in V(N)$ is the zero coalition function $v = 0$.

DEFINITION IV.20 (inessentiality). *A coalition function $v \in V(N)$ is called inessential if*

$$v(K) = \sum_{i \in K} v(\{i\})$$

holds for all $K \subseteq N$.

DEFINITION IV.21. *A coalition function is called additive if $v(R \cup S) = v(R) + v(S)$ holds for all coalitions R and $S \subseteq N$ obeying $R \cap S = \emptyset$.*

LEMMA IV.1. *A coalition function v is inessential if and only if every player $i \in N$ is an inessential player for v and if and only if v is additive.*

5.3. Monotonicity and superadditivity. Nearly all the coalition functions we work with in this book are monotonic (see definition IV.1 on p. 49) and superadditive. Monotonicity and superadditivity are closely related:

- Monotonicity means that adding players never decreases the worth.
- Superadditivity can be tanslated with "cooperation pays".

DEFINITION IV.22 (superadditivity). *A coalition function $v \in V(N)$ is called superadditive if for any two coalitons R and S*

$$R \cap S = \emptyset$$

implies

$$v(R) + v(S) \leq v(R \cup S).$$

$v(R \cup S) - (v(R) + v(S)) \geq 0$ *is called the gain from cooperation.*

Glove games are monotonic because the number of glove pairs cannot decrease if additional players (and hence additional gloves) are added. They are also superadditive because the number of glove pairs cannot decrease when two disjoint coalitions pool their gloves.

EXERCISE IV.17. *Is the coalition function v, given by $N = \{1, 2, 3\}$ and*

$$\begin{aligned} v(\{1,2,3\}) &= 5, \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 4, \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

superadditive?

EXERCISE IV.18. *How about superadditivity of unanimity games, of the Maschler game or of a contradictory simple game?*

While monotonicity and superadditivity seem very similar properties, monotonicity does not imply superadditivity as you can see from $N = \{1, 2\}$ and $v(\{1\}) = v(\{2\}) = 3$ and $v(\{1, 2\}) = 4$.

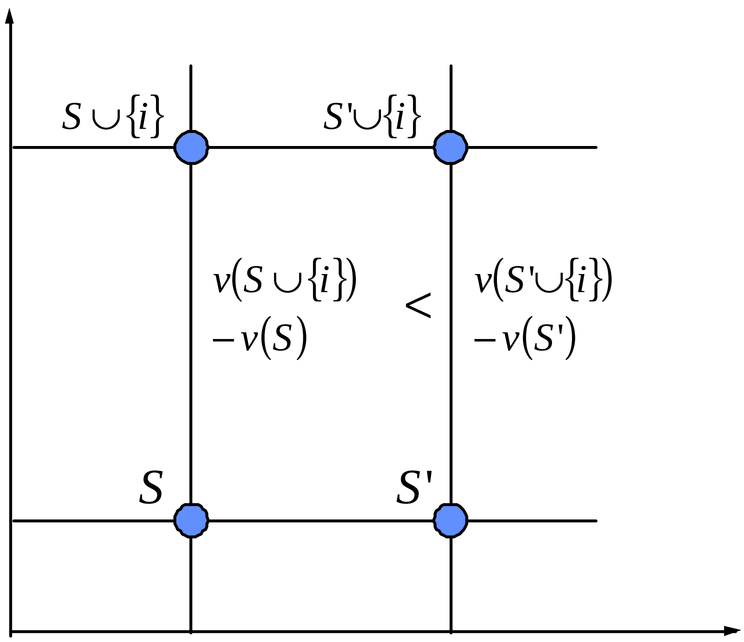

FIGURE 1. Strict convexity

EXERCISE IV.19. *Show that every monotonic game v is non-negative, i.e., fulfills $v(K) \geq 0$ for alle $K \subseteq N$.*

EXERCISE IV.20. *Show that superadditivity and non-negativity imply monotonicity.*

5.4. Convexity. Superadditivity means: cooperation pays. Convexity implies superadditivity, but is stronger. Convexity is interesting because the Shapley value can be shown to lie in the core of any convex game.

DEFINITION IV.23 (convexity). *A coalition function $v \in V(N)$ is called convex if for any two coalitons S and S' with $S \subseteq S'$ and for all players $i \in N \backslash S'$, we have*

$$v(S \cup \{i\}) - v(S) \leq v\left(S' \cup \{i\}\right) - v\left(S'\right).$$

v is called strictly convex if the inequality is strict.

Thus, the marginal contribution is large for large coalitions. May-be, you find fig. 1 helpful.

Let us consider the example of by $N = \{1,2,3,4\}$ and the coalition function v given by

$$v(S) = |S| - 1, S \neq \emptyset.$$

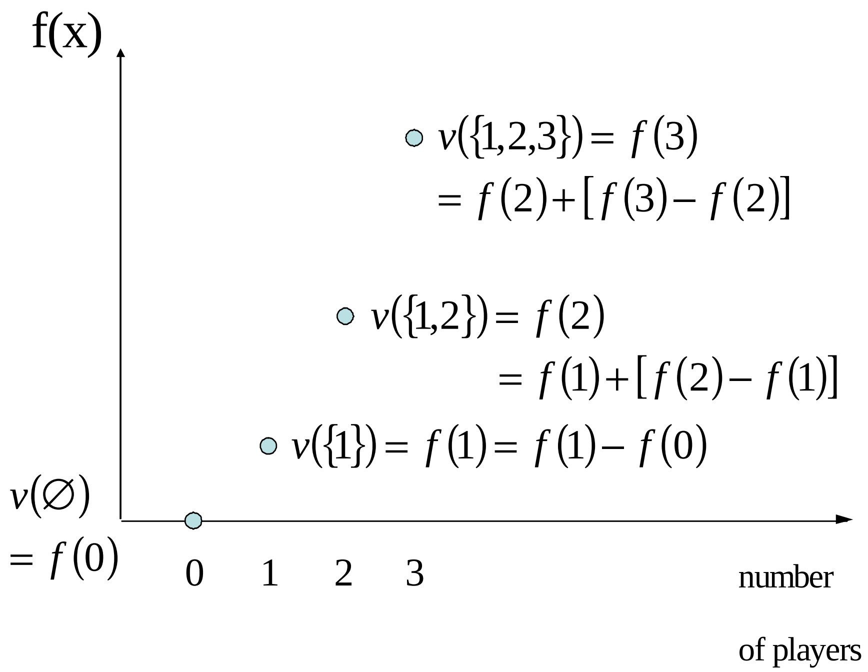

FIGURE 2. Convexity for symmetric coalition functions

Note that the marginal contribution is zero for any player who joins the empty set,

$$v(\emptyset \cup \{i\}) - v(\emptyset) = [|\{i\}| - 1] - 0 = 0,$$

while the marginal contribution with respect to any nonempty coalition is 1. Thus, this coalition function is convex.

EXERCISE IV.21. *Is the unanimity game u_T convex? Distinguish between $i \in T$ and $i \notin T$. Is u_T strictly convex?*

Why are convex coaliton functions called convex? The reader remembers that function $f : \mathbb{R} \to \mathbb{R}$ that are defined by $f(x) = x^2$ or $f(x) = e^x$ are called convex. If they are twice differentiable, the second derivatives (2 and e^x in our examples) are positive.

To see that convex coalition functions behave similarly, we consider the special case of symmetric coalition functions. In fig. 2, you see that the differences increase as they do for x^2.

Sometimes, an alternative characterization of convexity is helpful:

THEOREM IV.1 (criterion for convexity). *A coalition function v is convex if and only if for all coalitions R and S, we have*

$$v(R \cup S) + v(R \cap S) \geq v(R) + v(S).$$

v is strictly convex if and only if

$$v(R \cup S) + v(R \cap S) > v(R) + v(S)$$

holds for all coalitions R and S with $R \backslash S \neq \emptyset$ and $S \backslash R \neq \emptyset$.

We do not present a proof for this criterion. The reader can find a proof in the textbook on lattice theory by Topkis (1998).

We now turn to the relationship between superadditivity and convexity.

EXERCISE IV.22. *Is the Maschler game convex? Is it superadditive?*

Thus, a superadditive coalition function need not be convex. However, the inverse is true.

EXERCISE IV.23. *Using the above criterion for convexity, show that every convex coalition function is superadditive.*

5.5. The Shapley value and the core. The Shapley value need not be in the core even if the core is nonempty. This assertion follows from the following exercise that is taken from Moulin (1995, S. 425).

EXERCISE IV.24. *Consider the coalition function given by $N = \{1, 2, 3\}$ and*

$$v(K) = \begin{cases} 0, & |K| = 1 \\ \frac{1}{2}, & K = \{1,3\} \text{ or } K = \{2,3\} \\ \frac{8}{10}, & K = \{1,2\} \\ 1, & K = \{1,2,3\} \end{cases}$$

Show that $\left(\frac{4}{10}, \frac{4}{10}, \frac{2}{10}\right)$ belongs to the core but that the Shapley value does not.

However, the Shapley value can be shown to lie in the core for convex coalition functions:

THEOREM IV.2. *If a coalition function v is convex, the Shapley value $Sh(v)$ lies in the core.*

6. Topics and literature

The main topics in this chapter are

- simple game
- winning coalition
- veto player
- dictator
- null player
- unanimity game
- apex game
- weighted voting game
- buying-a-car game
- Maschler-Spiel
- endowment game
- superadditivity
- convexity
- monotonicity

We introduce the following mathematical concepts and theorems:

- linear independence
- span
- basis
- coefficients

We recommend the textbook by Wiese (2005).

7. Solutions

Exercise IV.1

$\eta_i = 0$ means that player i's marginal contribution is zero with respect to every coalition and hence player i is a null player. $\eta_i = 2^{n-1}$ implies that every subset K of $N\backslash\{i\}$ is a loosing coalition while $K \cup \{i\}$ is winning. Player i is a dictator and a veto player.

Exercise IV.2

Can there be a coalition K such that $v(K\backslash\{i\}) = 1$ for a veto player i or a dictator i?

If i is a veto player, we have $v(K\backslash\{i\}) \leq v(N\backslash\{i\}) = 0$ for every coalition $K \subseteq N$ and hence $v(K\backslash\{i\}) = 0$. Thus, a veto player $i \in N$ cannot fulfill $v(K\backslash\{i\}) = 1$. A dictator i cannot fulfill $v(K\backslash\{i\}) = 1$ because the worth of a coalition is 1 if and only if the dictator belongs to the coalition.

Exercise IV.3

A dictator is always a veto player – without him the coalition cannot win. However, a veto player need not be a dictator. Just consider the simple game v on the player set $N = \{1, 2\}$ defined by $v(\{1\}) = v(\{2\}) = 0$, $v(\{1, 2\}) = 1$. Players 1 and 2 are two veto players but not dictators.

Exercise IV.4

Let v be a simple game with a veto player $i \in N$. Then $v(K) = 1$ implies $i \in K$. By $i \notin N\backslash K$, we obtain $v(N\backslash K) = 0$ – the desired result.

Let v be a simple game with two veto players i and j, $i \neq j$. Then $v(\{i\}) = 0$ (by $j \notin \{i\}$) and $v(K\backslash \{i\}) = 0$ (by $i \notin K\backslash \{i\}$) hold.

Exercise IV.5

For the unanimity game u_T, the null players are the players from $N\backslash T$.

Exercise IV.6

The core is

$$\left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : x_1 + x_2 = 1\right\}$$

and the Shapley value is given by

$$Sh\left(u_{\{1,2\}}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right).$$

Exercise IV.7

For $n = 2$, we have

$$\begin{aligned} h_1(K) &= \begin{cases} 0, & K = \{1\} \text{ or } K = \emptyset \\ 1, & \text{otherwise} \end{cases} \\ &= u_{\{2\}}. \end{aligned}$$

$n = 3$ yields the symmetric game

$$h_1(K) = \begin{cases} 1, & |K| \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

(Symmetry means that the worths depend on the number of the players, only.)

Exercise IV.8

No, the apex player is not a veto player. If all the other player unite against the apex player, they win:

$$h_i(N\backslash \{i\}) = 1.$$

For the same reason, the apex player is not a dictator, either.

Exercise IV.9

We first show that h_i is not contradictory. Assume $h_i(K) = 1$ for any coalition $K \subseteq N$. Then, one of two cases holds. Either we have $K = N\backslash \{i\}$. This implies $h_i(N\backslash K) = h_i(\{i\}) = 0$. Or we have $i \in K$ and $|K| \geq 2$. Then, $h_i(N\backslash K) = 0$. Thus, h_i is noct contradictory.

We now show that h_i is decidable. Take any $K \subseteq N$ with $h_i(K) = 0$. This implies $K = \{i\}$ or $K \subsetneqq N\backslash \{i\}$. In both cases, the complements are winning coalitions: $N\backslash K = N\backslash \{i\}$ or $N\backslash K \supsetneqq \{i\}$.

Exercise IV.10

Since the apex player obtains the marginal contributions for positions 2 through $n-1$, his Shapley payoff is

$$\frac{n-2}{n}\cdot 1.$$

Due to efficiency, the other (symmetric!) players share the rest so that each of them obtains

$$\frac{1}{n-1}\left(1-\frac{n-2}{n}\right)=\frac{2}{n\left(n-1\right)}.$$

Thus, we have

$$Sh\left(h_1\right)=\left(\frac{n-2}{n},\frac{2}{n\left(n-1\right)},...,\frac{2}{n\left(n-1\right)}\right).$$

Exercise IV.11

One possible solution is

$$\left[1;\frac{1}{t},...,\frac{1}{t},0,...,0\right]$$

where $\frac{1}{t}$ is the weight for the powerful T-players while 0 is the weight for the unproductive $N\backslash T$-players.

Exercise IV.12

Every permanent member is a veto player by $4\cdot 7+10\cdot 1=38<39$. Because of $5\cdot 7+4\cdot 1=39$, four non-permanent members are necessary for passing a resolution.

Exercise IV.13

The voting rule is not contradictory and not decidable. This is just a corollary of exercise IV.4 (p. IV.4).

Exercise IV.14

By efficiency and symmetry, we have

$$Sh\left(v\right)=\left(24,24,24\right).$$

Exercise IV.15

The core has to fulfill

$$x_1+x_2+x_3+x_4=2$$

and also the inequalities

$$\begin{aligned} x_i &\geq 0, i=1,...,4,\\ x_1+x_3 &\geq 1,\\ x_1+x_4 &\geq 1,\\ x_2+x_4 &\geq 1 \text{ and}\\ x_2+x_3+x_4 &\geq 2.\end{aligned}$$

We then find

$$x_1=2-\left(x_2+x_3+x_4\right)\leq 0$$

and hence

$$\begin{aligned} x_1 &= 0 \text{ (because of } x_1 \geq 0\text{)}, \\ x_3 &\geq 1 \text{ and } x_4 \geq 1. \end{aligned}$$

Using efficiency once more supplies $x_2 = 0$ and

$$(0, 0, 1, 1)$$

is the only candidate for a core. Indeed, this is the core. Just check all the inequalities above and also those omitted. Player 2's payoff is 0 in this situation. If he burns his second glove, we find (non-generalized) gloves game $v_{\{1,2\},\{3,4\}}$ where player 2 may achieve any core payoff between 0 and 1.

Exercise IV.16

The number of gloves pairs in $v_{\mathcal{E}} \oplus v_{\mathcal{E}}$ is twice the number of glove pairs in $v_{\mathcal{E}}$.

Exercise IV.17

For any $i, j \in \{1, 2, 3\}$, $i \neq j$, we have $v(\{i\}) + v(\{j\}) = 0 + 0 < 4 = v(\{i, j\})$ and $v(\{i\}) + v(N \backslash \{i\}) = 0 + 4 < 5$. Hence, v is superadditive.

Exercise IV.18

Every unanimity game is superadditive. Assume a unanimity game u_T that is not superadditive. Then, we would have to disjunct coalitions R and S with $v(R) + v(S) > v(R \cup S)$. The whole set of productive players T cannot be contained in both R and S. If it is contained in R (or in S), it is also contained in $R \cup S$. Then, we have $v(R) + v(S) = 1 = v(R \cup S)$ and the desired contradiction. If T is not contained in R and not contained in S, we have $v(R) + v(S) = 0$ and the inequality cannot be true, either.

The Maschler game is also superadditive. We need to consider the two inequalities

$$0 + 0 \leq 60 \text{ and } 0 + 60 \leq 72.$$

A simple game is contradictory if we have a coaliton K such that $v(K) = v(N \backslash K) = 1$. By $v(K) + v(N \backslash K) = 2 > 1 = v(N)$, superadditivity is violated.

Exercise IV.19

For all coalitions $K \subseteq N$, we have $K \supseteq \emptyset$ and, by monotonicity $v(K) \geq v(\emptyset) = 0$.

Exercise IV.20

Consider two coalitions $S, S' \subseteq N$ with $S \subseteq S'$ gegeben. Monotonicity follows from

$$\begin{aligned} v(S') &= v\left(S \cup \left(S' \backslash S\right)\right) \\ &\geq v(S) + v\left(S' \backslash S\right) \text{ (superadditivity)} \\ &\geq v(S) \text{ (non-negativity)}. \end{aligned}$$

Exercise IV.21

Yes, u_T is convex. For $i \in T$ and $S \subseteq S' \subseteq N$ with $i \notin S'$, we obtain

$$\begin{aligned} u_T(S \cup \{i\}) - u_T(S) &= u_T(S \cup \{i\}) - 0 \ (S \not\supseteq T) \\ &\leq u_T\left(S' \cup \{i\}\right) - 0 \ (u_T \text{ is monotonic}) \\ &= u_T\left(S' \cup \{i\}\right) - u_T\left(S'\right) \ \ (S' \not\supseteq T). \end{aligned}$$

If, however, i is not included in T, both $v(S \cup \{i\}) - v(S)$ and $v(S' \cup \{i\}) - v(S')$ are equal to zero. This shows that u_T is convex, but not strictly convex.

Exercise IV.22

The Maschler game is superadditive (see exercise **IV.18, p. 60),** but not convex. For $S = \{1\}$, $S' = \{1,2\}$ and $i = 3$, we have

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= v(\{1,3\}) - v(\{1\}) = 60 \\ &> 12 = v(\{1,2,3\}) - v(\{1,2\}) \\ &= v\left(S' \cup \{i\}\right) - v\left(S'\right). \end{aligned}$$

Exercise IV.23

Let R and S be disjunct coalitions. If v is convex, we obtain

$$\begin{aligned} v(R \cup S) &= v(R \cup S) + v(\emptyset) \\ &= v(R \cup S) + v(R \cap S) \\ &\geq v(R) + v(S). \end{aligned}$$

Thus, v is superadditive.

Exercise IV.24

Player 3's Shapley value is

$$Sh_3(v) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{10} = \frac{7}{30}.$$

Symmetry and efficiency yield

$$Sh_1(v) = Sh_2(v) = \frac{1}{2} \cdot \left(1 - \frac{7}{30}\right) = \frac{23}{60}.$$

Since we have

$$Sh_1(v) + Sh_2(v) = 2 \cdot \frac{23}{60} = \frac{23}{30} < \frac{24}{30} = \frac{8}{10} = v(\{1,2\}),$$

the Shapley value does not belong to the core. You can check that $\left(\frac{4}{10}, \frac{4}{10}, \frac{2}{10}\right)$ fulfills all the necessary inequalities.

8. Further exercises without solutions

CHAPTER V

Dividends

In this chapter, we reconsider the vector space of coalition functions. It is a well-known result from linear algebra that every vector space has a basis. We discuss two different bases for the vector space of coalition functions.

1. Definition and interpretation

Harsanyi (1963) defines devidends:

Definition V.1 (Harsanyi dividend). *Let $v \in V(N)$ be a coalition function. The dividend (also called Harsanyi dividend) is a coalition function d^v on $V(N)$ defined by*

$$d^v(T) = \sum_{K \subseteq T} (-1)^{|T|-|K|} v(K).$$

Theorem V.1 (Harsanyi dividend). *For any coalition function $v \in V(N)$, its Harsanyi dividends are defined by the induction formula*

$$\begin{aligned} d^v(\emptyset) &= 0, \\ d^v(S) &= v(S) + \sum_{K \subset S} (-1)^{|S|-|K|} v(K) \end{aligned}$$

Why are the values of the coalition function d^v called dividends? Consider a player i who is a member of 2^{n-1} coalitions $T \subseteq N$. Player i "owns" coalition T together with the other players from T where his ownership fraction is $\frac{1}{|T|}$. Let us, now, assume that each coalition T brings forth a dividend $d^v(T)$. Then, player i should obtain the sum of average dividends

$$\sum_{i \in T \subseteq N} \frac{d^v(T)}{|T|}.$$

It can be shown that this sum equals the Shapley value $Sh_i(v)$. Thus, the term dividend makes sense if we assume that players get the Shapley value.

2. Coalition functions as vectors

As noted in chapter III, $V(N)$ can be considered the vector space of coalition functions on N. Since we have 2^n subsets of N, $2^n - 1$ (the worth of $\emptyset$ is always zero!) entries suffice to describe any game $v \in V(N)$. For example, $u_{\{1,2\}} \in G_{\{1,2,3\}}$ can be identified with the vector from $\mathbb{R}^7$

$$\left(\underbrace{0}_{\{1\}}, \underbrace{0}_{\{2\}}, \underbrace{0}_{\{3\}}, \underbrace{1}_{\{1,2\}}, \underbrace{0}_{\{1,3\}}, \underbrace{0}_{\{2,3\}}, \underbrace{1}_{\{1,2,3\}}\right).$$

EXERCISE V.1. *Write down the vector that describes the Maschler game*

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

You know how to sum vectors. We can also multiply a vector by a real number (scalar multiplication). Both operations proceed entry by entry:

EXERCISE V.2. *Consider* $v = (1,3,3)$, $w = (2,7,8)$ *and* $\alpha = \frac{1}{2}$ *and determine* $v + w$ *and* αw.

3. Spanning and linear independence

$\mathbb{R}^m$, $m \geq 1$, is a prominent class of vector spaces some of which obey $m = 2^n - 1$. We need some vector-space theory:

DEFINITION V.2 (linear combination, spanning). *A vector* $w \in \mathbb{R}^m$ *is called a linear combination of vectors* $v_1, ..., v_k \in \mathbb{R}^m$ *if there exist scalars (also called coefficients)* $\alpha_1, ..., \alpha_k \in \mathbb{R}$ *such that*

$$w = \sum_{\ell=1}^{k} \alpha_\ell v_\ell$$

holds. The set of vectors $\{v_1, ..., v_k\}$ *is said to span* $\mathbb{R}^m$ *if every vector from* $\mathbb{R}^m$ *is a linear combinations of the vectors* $v_1, ..., v_k$.

Consider, for example, $\mathbb{R}^2$ and the set of vectors

$$\{(1,2), (0,1), (1,1)\}.$$

Any vector (x_1, x_2) is a linear combination of these vectors. Just consider

$$\begin{aligned} & 2x_1 (1,2) - (3x_1 - x_2)(0,1) - x_1 (1,1) \\ = \; & (2x_1 - x_1, 4x_1 - (3x_1 - x_2) - x_1) \\ = \; & (x_1, x_2). \end{aligned}$$

EXERCISE V.3. *Show that* $(0,1)$ *is a linear combination of the other two vectors,* $(1,2)$ *and* $(1,1)$*!*

Using the result of the above exercise, we have

$$\begin{aligned} & 2x_1 (1,2) - (3x_1 - x_2)(0,1) - x_1 (1,1) \\ = \; & 2x_1 (1,2) - (3x_1 - x_2)[(1,2) - (1,1)] - x_1 (1,1) \\ = \; & [2x_1 - (3x_1 - x_2)](1,2) - [x_1 + (3x_1 - x_2)](1,1) \end{aligned}$$

so that any vector from $\mathbb{R}^2$ is a liner combination of just $(1,2)$ and $(1,1)$.

If we want to span $\mathbb{R}^2$ (or any $\mathbb{R}^m$), we try to find a minimal way to do so. Any vector in a spanning set that is a linear combination of other vectors in that set, can be eliminated.

DEFINITION V.3 (linear independence). *A set of vectors* $\{v_1, ..., v_k\}$ *is called linearly independent if no vector from that set is a linear combination of other vectors from that set.*

EXERCISE V.4. *Are the vectors* $(1,3,3), (2,1,1)$ *and* $(8,9,9)$ *linearly independent?*

Merging these two definitions gives rise to one of the most important concept for vector spaces.

DEFINITION V.4 (basis). *A set of vectors* $\{v_1, ..., v_k\}$ *is called a basis for* $\mathbb{R}^m$ *if it spans* $\mathbb{R}^m$ *and is linearly independent.*

An obvious basis for $\mathbb{R}^m$ consists of the m unit vectors

$$\begin{aligned} &(1,0,...,0), \\ &(0,1,0,...,), \\ &..., \\ &(0,...,0,1). \end{aligned}$$

Let us check whether they really do form a basis. Any $x = (x_1, ..., x_m)$ is a linear combination of these vectors by

$$\begin{aligned} & x_1(1,0,...,0) + x_2(0,1,0,...,) + ... + x_m(0,...,0,1) \\ = \ & (x_1,0,...,0) + (0,x_2,0,...,) + ... + (0,...,0,x_m) \\ = \ & (x_1,...,x_m). \end{aligned}$$

This proves that the unit vectors do indeed span $\mathbb{R}^m$.

In order to show linear independence, consider any linear combination of $m-1$ unit vectors, for example

$$\alpha_1(1,0,...,0) + \alpha_2(0,1,0,...,) + ... + \alpha_{m-1}(0,...,0,1,0)$$

which is equal to $(\alpha_1, ..., \alpha_{m-1}, 0)$ and unequal to $(0,...,0,1)$ for any coefficients $\alpha_1, ..., \alpha_{m-1}$.

LEMMA V.1 (basis of unit vectors). *The m unit vectors* $(1,0,...,0), ..., (0,...,0,1) \in \mathbb{R}^m$ *form a basis of the vector space* $\mathbb{R}^m$.

According to the above definition, a basis is a set of

(1) linearly independent vectors
(2) that span $\mathbb{R}^m$.

However, we do not need to check both conditions:

THEOREM V.2 (basis criterion). *Every basis of the vector space $\mathbb{R}^m$ has m elements. Any set of m elements of the vector space $\mathbb{R}^m$ that span $\mathbb{R}^m$ form a basis. Any set of m elements of the vector space $\mathbb{R}^m$ that are linearly independent form a basis.*

The reader might have noticed that the coefficients needed to express x as a linear combinations of unit vectors are uniquely determined. This is true for any basis:

THEOREM V.3 (uniquely determined coefficients). *Let $\{v_1, ..., v_m\}$ be a basis of $\mathbb{R}^m$ and let x be any vector such that*

$$x = \sum_{i=1}^{m} \alpha_i v_i = \sum_{i=1}^{m} \beta_i v_i.$$

Then $\alpha_i = \beta_i$ for all $i = 1, .., m$.

4. The basis of unanimity games

We have shown in the previous section that the unit games (that attribute the worth of one to exactly one nonempty coalition) form a basis of $V(N)$. They are the $2^n - 1$ coalition functions v_T, $T \neq \emptyset$, given by

$$v_T(S) = \begin{cases} 1, & S = T \\ 0, & S \neq T \end{cases}$$

An alternative and prominent basis of $V(N)$ is given by the unanimity games:

LEMMA V.2 (unanimity games form basis). *The $2^n - 1$ unanimity games u_T, $T \neq \emptyset$, form a basis of the vector space $V(N)$.*

According to theorem V.2, it is sufficient to show that the unanimity games are linearly independent. We use a proof by contradiction and assume that there is a unanimity game u_T that is a linear combination of the others:

$$u_T = \sum_{\ell=1}^{k.} \beta_\ell u_{T_\ell}$$

where

- the coalitions $T, T_1, ..., T_k$ are all pairwise different,
- $k \leq 2^n - 2$ holds and
- $\beta_\ell \neq 0$ holds for all $\ell = 1, ..., k$.

Let us assume $|T| \leq |T_\ell|$ for all $\ell = 1, .., k$. We can always rearrange the equation and rename the coalitions so that this condition is fulfilled. Using the coalition T as an argument, we now obtain

$$\begin{aligned} 1 &= u_T(T) \\ &= \sum_{\ell=1}^{k} \beta_\ell u_{T_\ell}(T) \\ &= \sum_{\ell=1}^{k} \beta_\ell \cdot 0 \\ &= 0 \end{aligned}$$

and hence the desired contradiction.

EXERCISE V.5. *In the above proof, do you see why* $u_{T_\ell}(T) = 0$ *holds for all* $\ell = 1, ..., k$*?*

Now, let us reconsider lemma V.2 and theorem V.3. They say that for any $v \in V(N)$ there exist uniquely determined coefficients $\lambda^v(T)$ such that

$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda^v(T) u_T$$

holds. This equation can also be expressed by

$$v(S) = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda^v(T) u_T(S), S \subseteq N. \tag{V.1}$$

Indeed, the coefficients can be shown to be the Harsanyi dividends:

$$\lambda^v(T) := d^v(T).$$

We will not provide a proof for this intriguing fact. Instead, we borrow an example from Slikker & Nouweland (2001, p. 7)). Consider $N := \{1, 2, 3\}$ and the coalition function v given by

$$v(S) = \begin{cases} 0, & |S| = 1 \\ 60, & S = \{1, 2\} \\ 48, & S = \{1, 3\} \\ 30, & S = \{2, 3\} \\ 72, & S = N \end{cases}$$

This coalition function can also be expressed by the vector

$$\begin{pmatrix} 0 \ (\{1\}) \\ 0 \ (\{2\}) \\ 0 \ (\{3\}) \\ 60 \ (\{1,2\}) \\ 48 \ (\{1,3\}) \\ 30 \ (\{2,3\}) \\ 72 \ (\{1,2,3\}) \end{pmatrix}$$

The coefficients are

$$\begin{aligned}
d^v(\{1\}) &= d^v(\{2\}) = d^v(\{3\}) = 0,\\
d^v(\{1,2\}) &= \sum_{K\in 2^{\{1,2\}}\backslash\{\emptyset\}} (-1)^{|\{1,2\}|-|K|}\, v(K)\\
&= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{2\}) + (-1)^{2-2} v(\{1,2\})\\
&= 0+0+60,\\
d^v(\{1,3\}) &= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{1,3\})\\
&= 48,\\
d^v(\{2,3\}) &= (-1)^{2-1} v(\{2\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{2,3\})\\
&= 30 \text{ and}\\
d^v(\{1,2,3\}) &= \sum_{K\in 2^{N}\backslash\{\emptyset\}} (-1)^{|N|-|K|}\, v(K)\\
&= (-1)^{3-1} v(\{1\}) + (-1)^{3-1} v(\{2\}) + (-1)^{3-1} v(\{3\})\\
&\quad + (-1)^{3-2} v(\{1,2\}) + (-1)^{3-2} v(\{1,3\}) + (-1)^{3-2} v(\{2,3\}) + (-1)^{3-3} v(\{1,2,3\})\\
&= 0+0+0-60-48-30+72\\
&= -66
\end{aligned}$$

and we obtain

$$\begin{aligned}
& d^v(\{1,2\})\, u_{\{1,2\}} + d^v(\{1,3\})\, u_{\{1,3\}} + d^v(\{2,3\})\, u_{\{2,3\}} + d^v(\{1,2,3\})\, u_N\\
= \;& 60\begin{pmatrix}0\\0\\0\\1\\0\\0\\1\end{pmatrix} + 48\begin{pmatrix}0\\0\\0\\0\\1\\0\\1\end{pmatrix} + 30\begin{pmatrix}0\\0\\0\\0\\0\\1\\1\end{pmatrix} - 66\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\end{pmatrix}\\
= \;& \begin{pmatrix}0\\0\\0\\60\\48\\30\\72\end{pmatrix}
\end{aligned}$$

and hence the expected vector.

Exercise V.6. *Calculate the coefficients for the following games on* $N = \{1,2,3\}$:

- $v \in V(N)$ *is defined by* $v(\{1,2\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$ *and* $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = 0$.

- *$v \in V(N)$ is defined by*

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ 8, & |S| = 2 \\ 9, & S = N \end{cases}$$

5. Topics and literature

The main topics in this chapter are

- Harsanyi dividend
- stability
- linear independence
- span
- basis
- coefficients

We recommend the textbook by Wiese (2005).

6. Solutions

Exercise V.1

The vector describing the Maschler game is

$$\left(\underbrace{0}_{\{1\}},\underbrace{0}_{\{2\}},\underbrace{0}_{\{3\}},\underbrace{60}_{\{1,2\}},\underbrace{60}_{\{1,3\}},\underbrace{60}_{\{2,3\}},\underbrace{72}_{\{1,2,3\}}\right).$$

Exercise V.2

We obtain $v+w=(1,3,3)+(2,7,8)=(3,10,11)$ and $\alpha w=\frac{1}{2}(2,7,8)=\left(1,\frac{7}{2},4\right)$.

Exercise V.3

We have $(1,2)-(1,1)=(0,1)$. Thus, we need the coefficients 1 and -1.

Exercise V.4

No, they are not linearly independent. Consider $2(1,3,3)+3(2,1,1)=(8,9,9)$.

Exercise V.5

Take any $\ell\in\{1,...,k\}$. In order for $u_{T_\ell}(T)=1$ to hold, T would need to be a superset of T_ℓ. However, by $|T|\leq|T_\ell|$, T and T_ℓ would then need to be equal which they are not.

Exercise V.6

In general, we have

$$d^v(T):=\sum_{K\in 2^T\setminus\{\emptyset\}}(-1)^{|T|-|K|}v(K).$$

For the first game, we find

$$\begin{aligned}
d^v(\{1\}) &= d^v(\{2\}) = d^v(\{3\}) = 0, \\
d^v(\{1,2\}) &= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{2\}) + (-1)^{2-2} v(\{1,2\}) = 1, \\
d^v(\{1,3\}) &= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{1,3\}) = 0, \\
d^v(\{2,3\}) &= (-1)^{2-1} v(\{2\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{2,3\}) = 1, \\
d^v(\{1,2,3\}) &= (-1)^{3-1} v(\{1\}) + (-1)^{3-1} v(\{2\}) + (-1)^{3-1} v(\{3\}) \\
&\quad + (-1)^{3-2} v(\{1,2\}) + (-1)^{3-2} v(\{1,3\}) + (-1)^{3-2} v(\{2,3\}) \\
&\quad + (-1)^{3-3} v(\{1,2,3\}) \\
&= 0 + 0 + 0 - 1 - 0 - 1 + 1 \\
&= -1
\end{aligned}$$

while the second leads to

$$\begin{aligned}
d^v(T) &= 0 \text{ für } |T| = 1, \\
d^v(T) &= d^v(\{1,2\}) = (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{2\}) + (-1)^{2-2} v(\{1,2\}) = 8 \text{ for } |T| = 2 \\
d^v(\{1,2,3\}) &= 3 \cdot (-1)^{3-2} v(\{1,2\}) + (-1)^{3-3} v(\{1,2,3\}) \\
&= -24 + 9 = -15.
\end{aligned}$$

7. Further exercises without solutions

CHAPTER VI

Axiomatizing the Shapley value

This *is* a book on applications. Nevertheless, the reader should see the most prominent example of the axiomatization of a value, the Shapley value. We presented the formula and the axioms in chapter III. Here, we want to provide the proof that formula and axioms are equivalent. Also, we show that several systems of axioms for the Shapley value exist. Finally, we present the Banzhaf solution which is an alternative to the Shapley value, in particular for simple games.

1. Introduction

The Shapley value is defined by

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^\rho(v).$$

This formula tells us to sum up and average the marginal contributions for each rank order. The formula obeys some axioms and disobeys others. We are interested in a particular set of axioms that is equivalent to the formula.

For any given set of axioms, we have three possibilities:

- There is no solution concept that fulfills all the axioms. That is, the axioms are contradictary.
- The axioms are compatible with several solution concepts.
- There is one and only one solution concept that fulfills the axioms. That is, the solution concept is axiomatized by this set of axioms.

DEFINITION VI.1. *A solution concept σ (on G) is said be axiomatized by a set of axioms if σ fulfills all the axioms and if any solution concept to do so is identical with σ.*

It turns out that the following four axioms do the trick for the Shapley value:

DEFINITION VI.2. *Let σ be a solution function σ. σ obeys*

- *the efficiency (or Pareto) axiom if $\sum_{i \in N} \sigma_i(v) = v(N)$ holds for all coalition functions $v \in G$,*
- *the symmetry axiom if $\sigma_i(v) = \sigma_j(v)$ is true for all coalition functions $v \in G$ and for any two symmetric players i and j,*
- *the null-player axiom if we have $\sigma_i(v) = 0$ for all coalition functions $v \in G$ and for any null player i and*

- *the additivity axiom in case of* $\sigma(v+w) = \sigma(v) + \sigma(w)$ *for any two coalition functions* $v, w \in G$ *with* $N(v) = N(w)$.

The main aim of this chapter is to prove

THEOREM VI.1 (1. axiomatization of Shapley value). *The Shapley formula is axiomatized by the four axioms mentioned in the previous definition.*

2. The Shapley formula fulfills the four axioms

2.1. Efficiency axiom. The efficiency axiom holds for the Shapley value and even for the marginal contributions.

DEFINITION VI.3 (ρ-solution). *For a player set* N *and a rank order* $\rho \in RO_N$, *the* ρ*-solution is given by*

$$\left(MC_1^\rho(v), ..., MC_n^\rho(v)\right).$$

Thus, let us assume any rank order $\rho \in RO_N$. We can savely assume $\rho = (1, ..., n)$. If the players come in a different order, we can rename them so as to obtain the order $(1, ..., n)$. We find

$$\begin{aligned}
\sum_{i\in N} MC_i^\rho(v) &= \sum_{i\in N} \left[v(K_i(\rho)) - v(K_i(\rho)\setminus\{i\})\right] \\
&= [v(\{\rho_1\}) - v(\emptyset)] \\
&\quad + [v(\{\rho_1,\rho_2\}) - v(\{\rho_1\})] \\
&\quad + [v(\{\rho_1,\rho_2,\rho_3\}) - v(\{\rho_1,\rho_2\})] \\
&\quad + ... \\
&\quad + \left[v\left(\{\rho_1,...,\rho_{n-1}\}\right) - v\left(\{\rho_1,...,\rho_{n-2}\}\right)\right] \\
&\quad + \left[v(\{\rho_1,...,\rho_n\}) - v\left(\{\rho_1,...,\rho_{n-1}\}\right)\right] \\
&= v(N) - v(\emptyset) \\
&= v(N).
\end{aligned}$$

LEMMA VI.1. *The* ρ*-solutions and the Shapley value fulfill the efficiency axiom.*

The efficiency of the ρ-solutions has been shown above. The efficiency of the Shapley value follows immediately:

$$\begin{aligned}
\sum_{i\in N} Sh_i(v) &= \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N} MC_i^{\rho}(v) \\
&= \sum_{\rho\in RO_N}\frac{1}{n!}\sum_{i\in N} MC_i^{\rho}(v) \text{ (rearranging the summands)} \\
&= \sum_{\rho\in RO_N}\frac{1}{n!} v(N) \text{ (ρ-solutions are efficient)} \\
&= n!\frac{1}{n!} v(N) \\
&= v(N).
\end{aligned}$$

2.2. Symmetry axiom. Astonishingly, the symmetry axiom is not easy to show. We refer the reader to Osborne & Rubinstein (1994, S. 293). Intuitively, symmetry is obvious. After all,

- two players are symmetric if they contribute in a similar fashion and
- the Shapley formula's inputs are these marginal contributions.

2.3. Null-player axiom. A null player contributes nothing, per definition. The average of nothing is nothing. Therefore, the null-player axiom holds for the Shapley value. Just look at

$$\begin{aligned}
\sum_{i\in N} Sh_i(v) &= \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N} MC_i^{\rho}(v) \\
&= \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N} 0 \\
&= 0.
\end{aligned}$$

2.4. Additivity axiom. In order to show additivity, note

$$\begin{aligned}
& (v+w)(K) - (v+w)(K\backslash\{i\}) \\
=\ & v(K) + w(K) - (v(K\backslash\{i\}) + w(K\backslash\{i\})) \\
=\ & [v(K) - v(K\backslash\{i\})] + [w(K) - w(K\backslash\{i\})]
\end{aligned}$$

for any two coalition functions $v, w \in V(N)$ any player $i \in N$ and any coalition $K \subseteq N$. Therefore, we find

$$\begin{aligned}
Sh_i(v+w) &= \sum_{i\in N} \frac{1}{n!} \sum_{\rho\in RO_N} MC_i^\rho (v+w) \\
&= \sum_{i\in N} \frac{1}{n!} \sum_{\rho\in RO_N} [(v+w)(K_i(\rho)) - (v+w)(K_i(\rho)\setminus\{i\})] \text{ (definition of marginal contr} \\
&= \sum_{i\in N} \frac{1}{n!} \sum_{\rho\in RO_N} ([v(K_i(\rho)) - v(K_i(\rho)\setminus\{i\})] + [w(K_i(\rho)) - w(K_i(\rho)\setminus\{i\})]) \text{ (see ab} \\
&= \sum_{i\in N} \frac{1}{n!} \sum_{\rho\in RO_N} [v(K_i(\rho)) - v(K_i(\rho)\setminus\{i\})] \\
&\quad + \sum_{i\in N} \frac{1}{n!} \sum_{\rho\in RO_N} [w(K_i(\rho)) - w(K_i(\rho)\setminus\{i\})] \\
&= Sh_i(v) + Sh_i(w).
\end{aligned}$$

3. ... and is the only solution function to do so

We now want to show that any solution function that fulfills the four axioms is the Shapley value. We follow the proof presented by Aumann (1989, S. 30 ff.). We remind the reader of two important facts.

- The unanimity games u_T, $T \neq \emptyset$, form a basis of the vector space $V(N)$ (see chapter IV, pp. 72) so that every coalition function v is a linear combination of these games:

$$v = \sum_{T\in 2^N\setminus\{\emptyset\}} \lambda_T(v)\, u_T. \tag{VI.1}$$

- For any game γu_T, $\gamma \in \mathbb{R}$, the players from $N\setminus T$ are the null players (compare exercise IV.5, S. 51).

Consider, now, any solution function σ that obeys the four axioms. We obtain

$$\begin{aligned}
\sum_{i\in T} \sigma_i(\gamma u_T) &= \sum_{i\in T} \sigma_i(\gamma u_T) + \sum_{i\in N\setminus T} \sigma_i(\gamma u_T) \text{ (null-player axiom)} \\
&= (\gamma u_T)(N) \text{ (Pareto axiom)} \\
&= \gamma u_T(N) \\
&= \gamma.
\end{aligned}$$

The null players (from $N\setminus T$) get zero payoff, the (symmetric!) T-players share γ :

$$\sigma_i(\gamma u_T) = \begin{cases} \frac{\gamma}{|T|}, & i \in T \\ 0, & i \notin T. \end{cases}$$

Let now v be any coalition function on N. Using the above results and applying the additivity axiom several times, we find

$$\begin{aligned}
\sigma_i(v) &= \sigma_i\left(\sum_{T\in 2^N\backslash\{\emptyset\}} \lambda_T(v)\,u_T\right) \text{ (eq. VI.1)} \\
&= \sum_{T\in 2^N\backslash\{\emptyset\}} \sigma_i(\lambda_T(v)\,u_T) \text{ (additivity axiom)} \\
&= \sum_{T\in 2^N\backslash\{\emptyset\}} \begin{cases} \frac{\lambda_T(v)}{|T|}, & i\in T \\ 0, & i\notin T. \end{cases} \quad (\text{with } \gamma := \lambda_T(v))
\end{aligned}$$

Thus, the axioms determine the payoffs. Since the Shapley formula fulfills the axioms, we obtain the desired result

$$\sigma = Sh.$$

And we are done.

4. A second axiomatization via marginalism

The Shapley value is an average of the marginal contributions of the players. Thus, whenever we have two coalition functions v and w such that the marginal contributions (with respect to any given coalition) of a player is the same under v and under w, the player's Shapley value is the same. This fact is called the marginalism axiom:

DEFINITION VI.4 (marginalism axiom). *A solution function σ is said to obey the marginalism axiom if, for any player $i\in N$ and any two coalition functions $v, w \in G$ with $N(v) = N(w)$,*

$$MC_i^K(v) = MC_i^K(w)\,, K\subseteq N(v)$$

implies

$$\sigma_i(v) = \sigma_i(w)\,.$$

The marginalism axiom is quite strong. Young (1985) has shown that the Shapley value can be axiomatized by just three axioms:

THEOREM VI.2 (2. axiomatization of Shapley value). *The Shapley formula is axiomatized by the symmetry axiom, the marginalism axiom and the efficiency axiom.*

5. A third axiomatization via balanced contributions

Finally, we want to consider the axiom of balanced contributions which is due to Myerson (1980). The basic idea is that players suffer equally if one of them withdraws from the game. We need some formal preliminary:

DEFINITION VI.5. *Let $v \in V(N)$ be a coalition function and let $S \subseteq N$, $S \neq \emptyset$ be a coalition. The restriction of v onto S is the coalition function*

$$\begin{aligned} v|_S &: 2^S \to \mathbb{R}, \\ K &\mapsto v|_S(K) = v(K). \end{aligned}$$

Thus, $v|_S$ attributes the same worths as v but only to subsets of S.

DEFINITION VI.6 (axiom of balanced contributions). *A solution function σ is said to obey the axiom of balanced contributions if, for any coalition function v and any two players $i, j \in N(v) =: N$,*

$$\sigma_i(v) - \sigma_i\left(v|_{N\backslash\{j\}}\right) = \sigma_j(v) - \sigma_j\left(v|_{N\backslash\{i\}}\right)$$

holds.

The reader notes that we employ the solution function on G, not on $V(N)$. After all, $v|_{N\backslash\{j\}}$ has one player less than game v. We will dwell on the interpretation of the balanced contributions in a minute. Before, let us note the axiomatization theorem:

THEOREM VI.3 (3. axiomatization of Shapley value). *The Shapley formula is axiomatized by the efficiency axiom and the axiom of balanced contributions.*

Balanced contributions is a very powerful axiom. Note, however, that we claim this axiom not just for a given player set N but for all its subsets also.

6. Balanced contributions and power-over

6.1. Introduction. The power of people and the power of some people over others have long been a central concern in sociology, politics, and psychology while Bartlett (1989) and Rothschild (2002) find a neglect of power apart from market power in mainstream economics. However, power seems to be an extraordinary elusive concept. As Bartlett (1989, pp. 9-10) observes, there exists a "multiplicity of concepts" of power, but no "widely accepted concept of power within either economics or its sister social sciences".

The thesis of this section is that there are basically three reasons for this lamentable state. First, power may be defined with reference to actions (actor 1 forces actor 2 to perform an act against 2's will) or with reference to payoffs (actor 1 benefits more than actor 2). This corresponds to the difference between I-power (with I standing for "influence") and P-power (with P denoting "prize" or "payoff") by Felsenthal & Machover (1998). Of course, I-power and P-power are closely related because actions result in payoffs and payoffs flow from actions.

An early and prominent definition of power is due to Max Weber (1968, p. 53):

> "Power is the probability that one actor within a social relationship will be in a position to carry out his own will despite resistance ... ."

Obviously, this is I-power. A Weberian P-power definition would be the following:

> "Power is the probability that one actor within a social relationship will obtain costly benefits from others."

Secondly, the multiplicity of power concepts also stems from the fact that power and power-over need to be distinguished. Consider James Coleman's (1990, p. 133) definition:

> "The power of an actor resides in his control of valuable events. The value of an event lies in the interests powerful actors have in that event. ... Power ... is not a property of the relation between two actors (so it is not correct in this context to speak of one actor's power over another, although it is possible to speak of the relative power of two actors)."

Most authors, however, prefer to understand power relatively, i.e., in terms of the power an actor 1 exercises over another actor 2. Proponents of this tradition are Max Weber (1968), Richard Emerson (1962), Dorwin Cartwright (1959, p. 196), and Vittorio Hösle (1997, p. 394-396) .

In this section, we will side with these authors and will talk about power in the sense of power-over. Our focus is on a third problem. According to some definitions, power is ubiquitous. For example, Viktor Vanberg (1982, p. 59, fn 48) observes that in every exchange relationship both sides do what they would not have done without the influence of the other party.

Indeed, if 1 offers 2 some money to perform a service and 2 obliges, does 1 have power over 2? Or, the other way around, does 2 have power over 1 because he "forces" 1 to give him money for some important (to 1) service. According to everyday usage, 1 exerts power over 2 if 1 obtains the service for "too little" money ("exploitation") while 2 exerts power over 1 if 2 asks for "too much" and 1 is in an urgent need for the service ("profiteering", "extortion", "usury").

In line with the above observation, we claim that every fruitful definition of power-over needs a reference point which may concern a "usual", "normal", or "moral" situation. We will argue for several and quite diverse reference points in section 6.2. It seems quite unavoidable that reference points contain some measure of arbitrariness and need to be defended rather specifically.

In section 6.3, we will try an alternative reference point that is not arbitrary. The idea of this reference point is simple. Actors may suffer (or gain) if other actors withdraw (where would you be without me?). In such a setting, 1 exerts power over 2 if 2 suffers more from a withdrawal by 1 than vice versa. However, we will find good reasons for this definition to fail. Indeed, if we use the Shapley value, withdrawal of 1 harms 2 as much as withdrawal of 2 harms 1 – this is the axiom of balanced contributions. While this may first seem counterintuitive, we will be able to indicate plausible mechanisms for this to come about.

The idea of this section is to tackle the reference-point issue by considering the difference between actual payoffs and payoffs according to some reference point. Of course, we will use cooperative game theory to define these payoffs.

The general idea of defining power by way of payoff differences can already be found in Johan Galtung (1969) who defines "violence ... as the cause of the difference between the potential and the actual". We will come back to Galtung's approach in section **??**. Less directly, Lukes (1986, p. 5) suggests "that to have power is to be able to make a difference to the world." Our difference approach captures these differences.

6.2. Payoff reflections of power-over .

6.2.1. *Payoff differences.* We want to measure power-over by looking at the payoff differences caused by the exercise of power of one player over another. In most examples, a player 1 exercises power over another player 2. We consider two coalition functions, v and w. Often, by v we mean a coalition function describing the actual social or economic situation where player 1 exercises power over player 2. w, on the other hand, describes what the players would get if, contrary to the actual state of affairs, player 2 were not subject to the power exerted by player 1. Formally, we usually get

$$D_1 := \varphi_1(v) - \varphi_1(w) > 0$$

and

$$D_2 := \varphi_2(v) - \varphi_2(w) < 0.$$

6.2.2. *Example: market power.* First, we consider the example of the gloves game where we assume one left-glove holder (player 1) and 4 right-glove holders (players 2 through 5). The left-glove holder is in a monopoly (or monopsony) position. The Shapley value is $\left(\frac{4}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$. Assume that player 1 sells his left glove. He obtains the price of $\frac{4}{5}$. Each of the players 2 through 5 have $\frac{1}{4}$ chance to buy the glove for a price of $\frac{4}{5}$. Hence, each right-glove holder has an expected utility of $\frac{1}{4}\left(1 - \frac{4}{5}\right) = \frac{1}{20}$.

Let us now invoke the norm of equal splitting of gains between player 1 and player 2 to whom player 1 happens to sell the left glove. Then, payoffs are $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. There exists a coalition function w leading to these payoffs.

Then, player 1's power over player 2 is reflected by

$$\begin{aligned} D_1 &= \varphi_1(v) - \varphi_1(w) \\ &= \frac{4}{5} - \frac{1}{2} \\ &= \frac{3}{10} \end{aligned}$$

and

$$\begin{aligned} D_2 &= \varphi_2(v) - \varphi_2(w) \\ &= \frac{1}{20} - \frac{1}{2} \\ &= -\frac{9}{20}. \end{aligned}$$

6.2.3. *Example: emotional dependence.* As a second example, we consider the emotional dependence that may sometimes exist between a player M (man) and a player W (woman). They may both like to live together so that $v(M,W) > 0$. However, he may be more independent of her than the other way around. Then,

$$v(M) > v(W)$$

is a plausible assumption. (If the reader finds the example objectionable, she or he is welcome to reverse the roles.)

The Shapley values are given by

$$\begin{aligned} \varphi_M &= \frac{1}{2}v(M) + \frac{1}{2}[v(M,W) - v(W)] \\ &= \frac{1}{2}v(M,W) + \frac{1}{2}[v(M) - v(W)] \\ &> \frac{1}{2}v(M,W) + \frac{1}{2}[v(W) - v(M)] \\ &= \varphi_W. \end{aligned}$$

His payoff is higher than her's. Applying the egalitarian norm ($w(M) = w(W) = \frac{1}{2}v(M,W)$) we obtain $\varphi_M(w) = \frac{1}{2}v(M,W) = \varphi_W(w)$. We would therefore diagnose that he has power over her:

$$\begin{aligned} D_M &= \varphi_M(v) - \varphi_M(w) \\ &= \frac{1}{2}[v(M) - v(W)] \\ &> 0 \\ &> \frac{1}{2}[v(W) - v(M)] \\ &= D_W \end{aligned}$$

Both examples make clear that the problem about a reference point is not "solved". We rather choose to offer a taxonomy: If the reference point is some or other norm (or defined by some or other counterfactual), then we obtain this or that payoff difference. While this may seem an evasive strategy, we argue that power-over necessarily needs a reference point and that there is no unambiguous choice of such a point.

6.3. Action reflexions of power-over.

6.3.1. *Withdrawing and quitting.* Instead of invoking some quite arbitrary fairness norms, one might consider the differences

$$\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right)$$

and

$$\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right)$$

known from the axiom of balanced contributions. For player 1, $v|_{N\backslash 2}$ is the game v without player 2. In words: $\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right)$ measures the loss to player 1 if player 2 withdraws. We might try the following definition: Player 1 exerts power over player 2, if player 1 suffers less from a withdrawal by player 2 than vice versa.

Interestingly, this definition fails if we use the Shapley value: What 1 can do to 2 by withdrawing is exactly equal to what 2 can do to 1 by withdrawing. This is just what balanced contributions means.

6.3.2. *Example: Revisiting the gloves game.* Let us reconsider the gloves game. Again, we assume one left-glove holder (player 1) and 4 right-glove holders (players 2 through 5) (see subsection 6.2.2). It might seem that player 1's threat of withdrawal carries more weight than player 2's threat of withdrawal. However, this is not the case. The Shapley values are

$$\begin{aligned} \left(\frac{4}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right) \text{ for } N &= \{1,2,3,4,5\}, \\ \left(\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right) \text{ for } N &= \{1,3,4,5\} \text{ and} \\ (0,0,0,0) \text{ for } N &= \{2,3,4,5\} \end{aligned}$$

so that we have

$$\begin{aligned} &\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right) \\ = &\frac{4}{5} - \frac{3}{4} \\ = &\frac{1}{20} \end{aligned}$$

and

$$\begin{aligned} &\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right) \\ = &\frac{1}{20} - 0. \end{aligned}$$

The reason for the equality of these differences is this: Player 1 obtains a price of $\frac{4}{5}$ for his left glove in case of 4 potential buyers, but a price of $\frac{3}{4}$ in case of 3 potential buyers. So indeed, player 2's withdrawal does not do much damage to player 1. But player 2's disutility caused by player 1's withdrawal is small also. If player 1 is around, player 2 will have a small chance ($\frac{1}{4}$) of getting the glove and will also have to pay a high price ($\frac{4}{5}$). Therefore, in the presence of player 1, player 2 gets the payoff $1 - \frac{4}{5} = \frac{1}{5}$ with a chance of $\frac{1}{4}$ only. The small payoff of 1/20 is lost when player 1 withdraws.

While payoff differences with respect to the threat of withdrawal are not useful for defining power-over, they can be used to theorize about the action players have to take. In the gloves example, it is the balanced contributions that allow player 1 to charge a high price for his left glove.

6.3.3. *Example: Revisiting emotional dependence.* We also reconsider the emotional-dependence example (see section 6.2.3) and obtain her payoff difference as

$$\begin{aligned} & \varphi_W(v) - \varphi_W\left(v|_{N\setminus M}\right) \\ = & \left[\frac{1}{2}v(M,W) + \frac{1}{2}v(W) - \frac{1}{2}v(M)\right] - v(W) \\ = & \frac{1}{2}\left[v(M,W) - v(W) - v(M)\right]. \end{aligned}$$

In case of superadditivity, his threat of withdrawal (divorce, say) is effective and she suffers from it. However, for player M we get the same result:

$$\varphi_M(v) - \varphi_M\left(v|_{N\setminus W}\right) = \varphi_W(v) - \varphi_W\left(v|_{N\setminus M}\right).$$

Again, we can use this equality to infer actions: Just because of $v(M) > v(W)$, he can make her do the washing-up. But taking her washing-up into account, she suffers less from a break-down of the relationship and his loss of her would be more serious than in a "fair" partnership.

6.4. Negative sanctions and the threat to withdraw. The equality of the threats to withdraw may be particularly astonishing for negative sanctions and coercion (see Willer 1999, pp. 24). Indeed, if a robber (player 1) points his gun to my, player 2's, head, it may seem impossible for me to "withdraw". However, we need to look more closely.

It is important to note that withdrawing is analyzed within the given game v. The question of whether a player can quit a game or opt out is a totally different one. For example, I normally do not need to partake in a market game but sometimes I cannot help being part of a game as in our gun-and-money game.

First, we need to define the coalition function. For the coalition $\{1,2\}$, $v(1,2) = 0$ seems plausible. I hand over some money $c > 0$ to the robber so that his gain is my loss. We then have $\varphi_1(v) = c = -\varphi_2(v)$ which fulfills the efficiency axiom. (Of course, I may be traumatized by the experience and he may be afraid of being caught and arrested in which case $v(1,2)$ should be negative.)

One may be tempted to put $v(2) = 0$ since I do not lose any money if the robber is not there. However, what I can achieve on my own still depends on what the robber does (withdrawal is not quitting!). If I do not hand over the money peacefully, he may injure me. We define the worth for a coalition K as the minimum of what the other players, $N \backslash K$, can inflict on K. We let i represent the pain of being injured and obtain $v(2) = -i < 0$.

Similarly, $v(1)$ is the minimum of what I can inflict on the robber. I can run away and force him to injure me. Then, he will be in fear of prosecution for injury; let f stand for this fear so that we have $v(1) = -f$.

Now, because of $v(1) = v|_{N\backslash 2}(1)$ and $\varphi_1\left(v|_{N\backslash 2}\right) = v(1)$, my running away or his injuring me leads to the payoff differences

$$\begin{aligned} &\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right) \\ = &\underbrace{c}_{\substack{\text{money}\\ \text{obtained}}} - \underbrace{-f}_{\substack{\text{disutility from fear of}\\ \text{prosecution for injury}}} \end{aligned}$$

and

$$\begin{aligned} &\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right) \\ = &\underbrace{-c}_{\substack{\text{money given}\\ \text{to robber}}} - \underbrace{-i}_{\text{disutility from injury}} \end{aligned}$$

The equality between these two differences can now be used to calculate the money I will have to hand over to the robber. It is given by

$$c = \frac{i-f}{2}.$$

The less the robber's fear of prosecution for injury and the higher my unwillingness to suffer injury, the higher the robber's loot. For c to be non-negative, we need $i \geq f$; my fear of injury has to be higher than the robber's fear of prosecution.

6.5. Revisiting Weber's definition of power. For the Shapley value, the threat of withdrawal from a cooperative agreement has to be symmetric between the two players. In the gloves game, this symmetry determines the price of gloves; in the emotional-dependence example it leads to her doing the washing up; and in the case of robbery, the robber's gain obtains.

Of course, the holder of the non-scarce commodity would prefer a fair price of $\frac{1}{2}$, the dependent woman would like to share the burden of housework evenly, and the victim of robbery would prefer to hold on to his money. However, the holder of the scarce commodity, the man in the dependency example and the robber manage to "realize their own will ... against the resistance" of the other party. We just cited Max Weber in order to indicate that we consider these three examples instances of power in his sense.

In fact, a research program suggests itself: Whenever we have a seemingly asymmetric power-over relationship we should look out for Weberian power by equalizing the payoff differences with respect to the threat of withdrawal. For example, power-over relationships may exist between parents and children, God and humans, a king and his subjects, a bureaucrat and people obtaining permission, master and slave, etc.. Which actions lead to balanced contributions?

7. The Banzhaf solution

7.1. The Banzhaf formula. The Banzhaft solution is due to Banzhaf (1965) who applied it to weighted majority games. The Banzhaf formula is given by

$$Bh_i(v) = \frac{1}{2^{n-1}} \sum_{\substack{K \subseteq N, \\ i \notin K}} [v(K \cup \{i\}) - v(K)], i \in N.$$

Similar to the Shapley value, an average of marginal contributions is calculated. However, while Shapley considers all rank orders, Banzhaf proposes to look at all coalitions which (do not) contain a given player i. We can find

$$\left|2^{N\backslash\{i\}}\right| = 2^{|N\backslash\{i\}|} = 2^{n-1}$$

of these coalitions.

Thus, under the Shapley value, every rank order has the same probability while the Banzhaf index attributes the same probability for each coalition that contains a specfic player.

Exercise VI.1. *Given $N = \{1, 2, 3\}$, write down the coalitions that do not contain player i.*

The Banzhaf formula can be applied to any game but the main field of application concerns simple games. Then, the Banzhaf formula is also called Banzhaft power index or Banzhaf index.

Restricting attention to simple games, we can focus on pivotal coalitions. We remind the reader of the definition found in chapter IV:

Definition VI.7 (pivotal coalition). *For a simple game v, $K \subseteq N$ is a pivotal coalition for $i \in N$ if $v(K) = 0$ and $v(K \cup \{i\}) = 1$. The number of i's pivotal coalitions is denoted by $\eta_i(v)$,*

$$\eta_i(v) := |\{K \subseteq N : v(K) = 0 \text{ and } v(K \cup \{i\}) = 1\}|.$$

We have $\eta(v) := (\eta_1(v), ..., \eta_n(v))$ and $\bar{\eta}(v) := \sum_{i \in N} \eta_i(v)$. We sometimes omit the game and write η_i (η, $\bar{\eta}$) rather than $\eta_i(v)$ ($\eta(v), \bar{\eta}(v)$).

Thus, a player i is pivotal for a coalition K if $v(K)=0$ and $v(K\cup\{i\})=1$ hold. Player i's number of pivotal coalitions is denoted by $\eta_i(v)$ (or η_i).

EXERCISE VI.2. *Find η_i for a null player and for a dictator.*

Now, the Banzhaf index for player i can be rewritten as

$$Bh_i(v)=\frac{\eta_i}{2^{n-1}}.$$

EXERCISE VI.3. *Calculate the Banzhaf payoffs for player* 1 *in case of* $N=\{1,2,3\}$ *and* $u_{\{1,2\}}$. *What do you find for* $N=\{1,2,3,4\}$ *and* $u_{\{1,2,3\}}$*?*

EXERCISE VI.4. *Find the Banzhaf payoffs for* $N=\{1,2,3,4\}$ *and the apex game* h_1 *defined by*

$$h_1(K)=\begin{cases}1, & 1\in K \text{ and } K\backslash\{1\}\neq\emptyset\\ 1, & K=N\backslash\{1\}\\ 0, & \text{sonst}\end{cases}$$

Does the Banzhaf solution fulfill Pareto efficiency?

7.2. The Banzhaf axiomatization. While the Banzhaf index violates Pareto efficiency in general, it always fulfills the other three Shapley axioms. Indeed, the following theorem can be shown:

THEOREM VI.4 (axiomatization of the Banzhaf value). *The Banzhaf formula is axiomatized by null-player axiom, the symmetry axiom, the marginalism axiom and the merging axiom.*

You know all these axioms except the merging axiom. It means that if you merge two players into one player, then this new player obtains the sum of what the two constituent players got.

DEFINITION VI.8 (merging players). *For a game* (N,v) *and two players* $i,j\in N$, $i\neq j$, *the merged game* (N_{ij},v_{ij}) *is given by* $N_{ij}=(N\backslash\{i,j\})\cup\{ij\}$ *and*

$$v_{ij}(K)=\begin{cases}v(K), & K\subseteq N\backslash\{ij\}\\ v((K\backslash\{ij\})\cup\{i,j\}), & ij\in K\end{cases}$$

for all $K\subseteq N_{ij}$.

DEFINITION VI.9 (merging axiom). *A solution function* σ *is said to obey the merging axiom if we have*

$$\sigma_i(v)+\sigma_j(v)=\sigma_{ij}(N_{ij},v_{ij})$$

for any merged game in the sense of the definition above.

Consider the gloves game $v_{\{1,2\},\{3\}}$. Its Shapley payoffs are $Sh\left(v_{\{1,2\},\{3\}}\right)=\left(\frac{1}{6},\frac{1}{6},\frac{2}{3}\right)$ while the Banzhaf formula yields $Bh\left(v_{\{1,2\},\{3\}}\right)=\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)$.

Let us now assume that players 1 and 2 merge. The new player 12 obtains the Shapley payoff $\frac{1}{2}>\frac{1}{6}+\frac{1}{6}$. Intuitively, he players 1 and 2 (from the same market side) do not compete against each other any more so that their joint payoff increases while player 3 suffers. In contrast the Banzhaf payoffs are $\frac{1}{2}$ for both 12 and 3. In line with the merging axiom, we have $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. However, player 3's payoff reduces so that there is some indication of decreased competition between the left-hand glove owners even for this value.

If players 2 and 3 merge, the new player 23 is a dictator with Shapley value 1 and Banzhaf value 1. Again, the Banzhaf value obeys the merging axiom while the Shapley value does not.

8. Topics and literature

The main topics in this chapter are

- axiomatization
- balanced contributions
- marginalism
- power-over
- P-power and I-power

We introduce the following mathematical concepts and theorems:

- t

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We recommend the textbook by Wiese (2005).

9. Solutions

Exercise VI.1

Player 1 does not belong to four coalitions: $\emptyset, \{2\}, \{3\}, \{2,3\}$.

Exercise VI.2

For a null player, we find $\eta_i = 0$, while $\eta_i = 2^{n-1}$ characterizes a dictator.

Exercise VI.3

Player 1 has the two pivotal coalitions, $\{2\}$ and $\{3\}$. Therefore, his Banzhaf index is $\frac{2}{4} = \frac{1}{2}$.

Exercise VI.4

For player 1, every coalition is pivatal except $\emptyset$ and $\{2,3,4\}$. Therefore, we find $Bh_1(h_1) = \frac{6}{8} = \frac{3}{4}$.

Player 2's pivatal coalitions are $\{1\}$ and $\{2,3\}$ and he therefore obtains $Bh_2(h_1) = \frac{2}{8} = \frac{1}{4}$. By symmetry, we obtain $Bh_3(h_1) = Bh_4(h_1) = \frac{1}{4}$. Therefore, the sum of Banzhaf payoffs exceeds the worth of the grand coalition:

$$\frac{3}{4} + 3 \cdot \frac{1}{4} = \frac{3}{2} > 1 = h_1(N).$$

The Banzhaf index is not Pareto efficient.

10. Further exercises without solutions

(including Banzhaf)

The Shapley value on partitions

The Shapley value on networks

Giving voluntarily and taking by force

Extensions and vector-measure games

Non-transferable utility

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